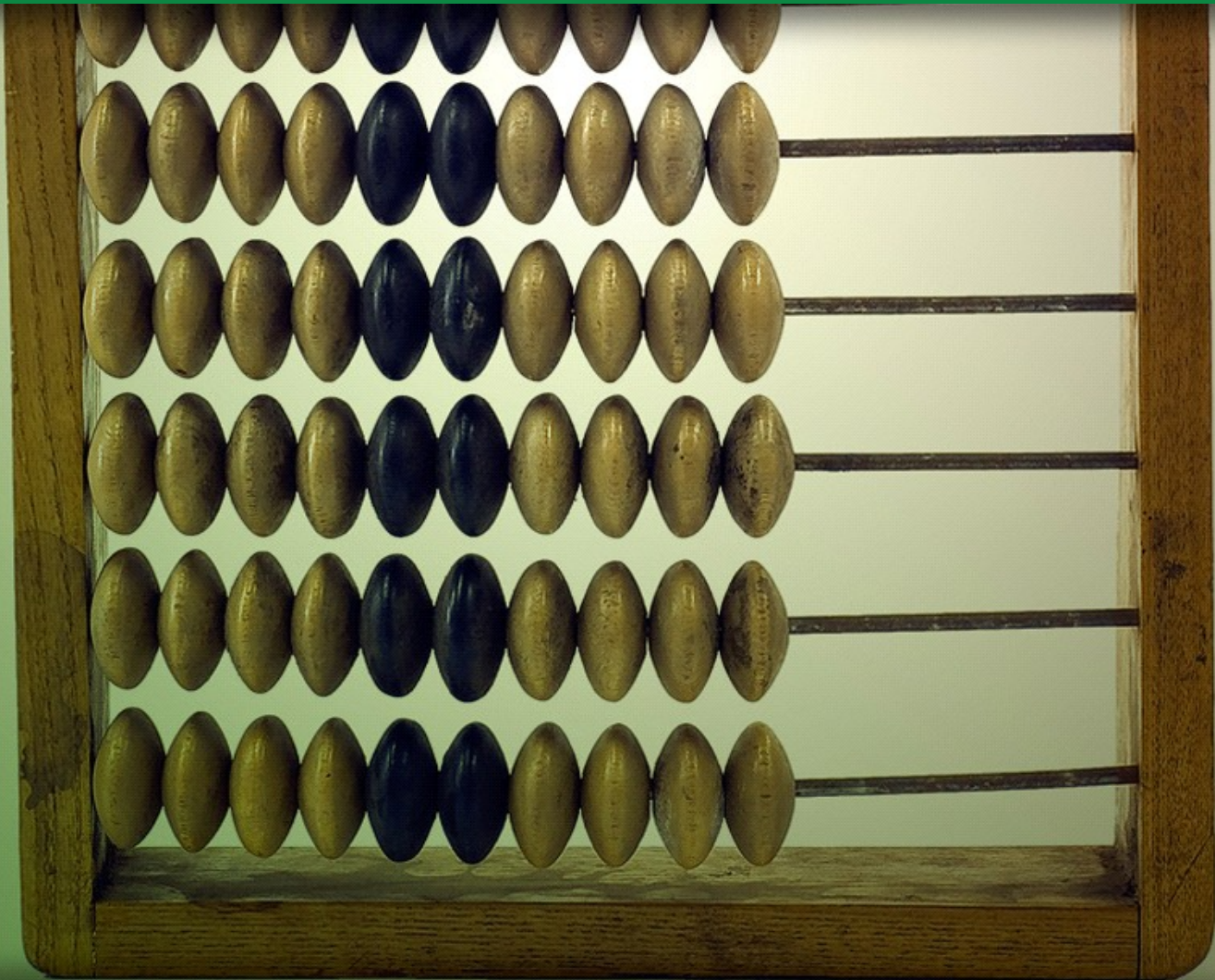


**LEIF MEJLBRO**

**REAL FUNCTIONS OF SEVERAL VARIABLES -  
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# Real Functions in Several Variables

Examples of Maximum and Minimum Integration  
and Vector Analysis

Calculus 2b

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Calculus 2b – Real Functions in Several Variables  
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# 1 Preface

The purpose of this volume is to present some worked out examples from the theory of Functions in Several Variables in the following topics:

- 1) Maximum and minimum of a function.
- 2) Integration in the plane and in the space.
- 3) Vector analysis.

As an experiment I shall here use the following generic diagram for solving problems:

**A.** For **Awareness**. What is the problem?

Try to formulate the problem in your own words, thereby identifying it.

**D.** For **Decision**. What are we going to do with it?

Are there any reasonable solution procedure available? If so, which one should be chosen?

**I.** For **Implementation**. Here we do all the necessary calculations for solving the task after the choice of the previous **D**.

At high school one usually starts here, but the problems may now be so complex that we need the previous analysis as well.

**C.** For **Control**. Whenever it is possible, one should check the solution.

Note, however, that this is not always possible, so in many cases we have to skip this point.

Notice that **A**, **D**, **I** can always be effectuated, no matter whether the problem is a mathematical exercise, or construction of some building, or any other problem which should be solved. The model is in this sense generic. It was first presented for me in *Telecommunication* for over 15 years ago, where I added **C**, the control of the solution. I hope that these simple guidelines will help the students as much as it has helped me. Notice also that if one during the **I**, *Implementation*, comes across a new and unforeseen problem, then one may iterate this simple model.

The intension is *not* to write a textbook, but only instead to give some hints of how to solve problems in this field. It therefore cannot replace any given textbook, but it may be used as a supplement to such a book on Functions in Several Variables.

The chapters are only consisting of examples without any further mathematical theory, which one must get from an ordinary textbook. On the other hand, it should be possible to copy the methods given here in similar exercises.

In Appendix A the reader will find a collection of formulæ which otherwise tacitly are assumed to be known from high school. It is highly recommended that the student learns these by heart during the course, because they form the backbone of the elementary part of Calculus, which should be mastered, before one may proceed to more advanced parts of Mathematics.

The text is a continuation of Studentensupport: Calculus 1, Real Functions in One Variable and of Studentensupport: Calculus 2a, Real Functions in Several Variables, Methods of Solution.

The text is based on my experiences in my teaching of students in this course. I realized that there was absolutely a need for a practical description of how to solve explicit problems.

Leif Mejlbro

## 2 The range of a function in several variables

### 2.1 Maximum and minimum

#### Example 2.1

**A** Let  $A$  be a closed and bounded (i.e. compact) subset of the plane where the boundary  $\partial A$  is a closed curve of the parametric representation

$$\mathbf{r}(t) = \left( 4t^{\frac{1}{3}}(1-t)^{\frac{2}{3}}, 4t^{\frac{2}{3}}(1-t)^{\frac{1}{3}} \right), \quad t \in [0, 1].$$

Find the maximum and minimum in  $A$  for the  $C^\infty$ -function

$$f(x, y) = x^3 + y^3 - 3xy, \quad (x, y) \in A.$$

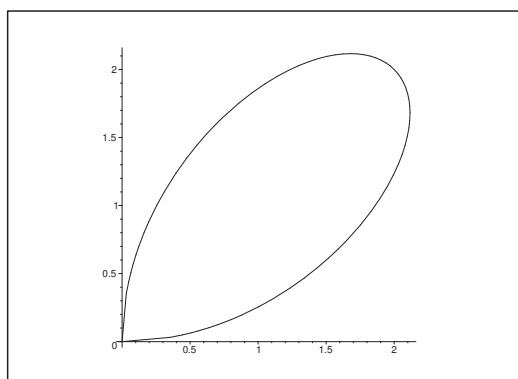


Figure 1: The closed and bounded domain  $A$ .

#### D Standard procedure:

- 1) Sketch the *domain*  $A$  and apply the *second main theorem for continuous functions*, from which we conclude the existence of a maximum and a minimum.
- 2) Identify the *exceptional points* in  $A^\circ$ , if any, and calculate the values  $f(x, y)$  in these points.
- 3) Set up the equations for the *stationary points*; find these – which quite often is a fairly difficult task, because the system of equations is usually non-linear. Finally, compute the values  $f(x, y)$  in all stationary points.
- 4) Examine the function on the boundary, i.e. restrict the function  $f(x, y)$  to the boundary and repeat the investigation above to a set which is of lower dimension. Then find the maximum and minimum on the boundary.
- 5) Collect all the *candidates* for a maximum and a minimum found previously in 2)–4). Then the maximum  $S$  and the minimum  $M$  are found by a simple numerical comparison.

**Remark 2.1** Note that by using this method there is no need to use the complicated  $(r, s, t)$ -method, which only should be applied when we shall find *local* extrema in the plane. Here we are dealing with *global* maxima and minima in a set  $A$ .  $\diamond$

**Remark 2.2** Sometimes it is alternatively easy to identify the *level curves*  $f(x, y) = c$  for the function  $f$ . In such a case, sketch a convenient number of the level curves, from which it may be easy to find the largest and the smallest constant  $c$ , for which the corresponding level curve has points in common with the set  $A$ . Then these values of  $c$  are automatically the maximum  $S$ , resp. the minimum  $M$  for  $f$  on  $A$ .

Notice, however, that this alternative method is demanding some experience before one can use it as a standard method of solution. It was once used with success by a brilliant student at an examination, summer 2000.  $\diamond$

**I** The level curves  $f(x, y) = x^3 + y^3 - 3xy = c$  do not look to promising, so we stick to the standard procedure.

- 1) The domain  $A$  has already been sketched. Since  $A$  is *closed and bounded*, and  $f(x, y)$  is *continuous* on  $A$ , it follows from the *second main theorem for continuous functions* that the function  $f$  has a maximum and a minimum on the set  $A$ .
- 2) Since  $f$  is of class  $C^\infty$  in  $A^\circ$ , there are no *exceptional points*.

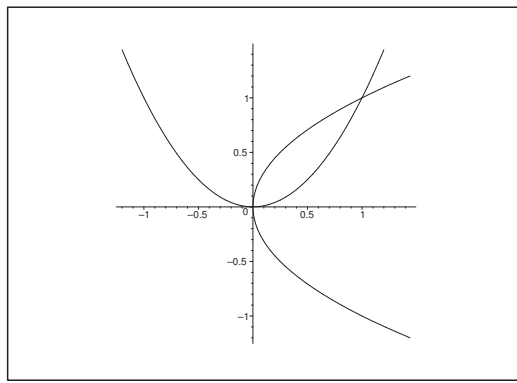




Figure 2: The stationary points are the intersections between the curves  $y = x^2$  and  $x = y^2$ .

3) The *stationary points* satisfy the two equations

$$\frac{\partial f}{\partial x} = 3x^2 - 3y = 0, \quad \text{i.e.} \quad y = x^2,$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3x = 0, \quad \text{i.e.} \quad x = y^2.$$

When we look at the *graph* we obtain the two solutions:

$$(0, 0) \in \partial A \quad \text{and} \quad (1, 1) \in A^\circ.$$

**Alternatively** one inserts  $y = x^2$  into the second equation

$$0 = y^2 - x = x^4 - x = x(x^3 - 1) = x(x - 1)(x^2 + x + 1).$$

Here  $x^2 + x + 1$  has only *complex* roots, hence the only *real* roots are  $x = 0$  (with  $y = x^2 = 0$ ) and  $x = 1$  (with  $y = x^2 = 1$ ), corresponding to

$$(0, 0) \in \partial A \quad \text{and} \quad (1, 1) \in A^\circ.$$

Since  $(0, 0)$  is a *boundary point*, we see that  $(1, 1) \in A^\circ$  is the *only stationary point* for  $f$  in  $A^\circ$ .

We transfer the value

$$f(1, 1) = 1 + 1 - 3 = -1.$$

to the collection of all values in 5) below.

4) *The Boundary.* When we apply the parametric representation

$$(x, y) = \mathbf{r}(t), \quad t \in [0, 1],$$

we get the *restriction* to the boundary

$$\begin{aligned} g(t) &= f(\mathbf{r}(t)) = f\left(4t^{\frac{1}{3}}(1-t)^{\frac{2}{3}}, 4t^{\frac{2}{3}}(1-t)^{\frac{1}{3}}\right) \\ &= \{64t(1-t)^2\} + \{64t^2(1-t)\} - 3 \cdot \left\{4t^{\frac{1}{3}}(1-t)^{\frac{2}{3}}\right\} \cdot \left\{4t^{\frac{2}{3}}(1-t)^{\frac{1}{3}}\right\} \\ &= 64t(1-t)^2 + 64t^2(1-t) - 48t(1-t) \\ &= 16t(1-t)\{4(1-t) + 4t - 3\} = 16t(1-t), \quad t \in [0, 1]. \end{aligned}$$

We have now reduced the problem to a problem known from high school

$$g'(t) = 12(2t - 1) = 0 \quad \text{for } t = \frac{1}{2},$$

corresponding to

$$g\left(\frac{1}{2}\right) = f\left(4 \cdot 2^{-\frac{1}{3}} \cdot 2^{-\frac{2}{3}}, 4 \cdot 2^{-\frac{2}{3}} \cdot 2^{-\frac{1}{3}}\right) = f(2, 2) = 4.$$

At the end points of the interval,  $t = 0$  and  $t = 1$ , we get

$$g(0) = g(1) = f(0, 0) = 0.$$

5) We collect all the candidates:

exceptional points: None, [from 2)]

Stationary point:  $f(1, 1) = -1$ , [from 3)]

Boundary points:  $f(0, 0) = 0$  and  $f(2, 2) = 4$ , [from 4)].

By a *numerical comparison* we get

- The *minimum* is  $f(1, 1) = -1$  (a stationary point),
- The *maximum* is  $f(2, 2) = 4$  (a boundary point).

6) A typical addition: Since  $A$  is *connected*, and  $f$  is *continuous*, it also follows from the *first main theorem for continuous functions*, that the range is an interval (i.e. connected), hence

$$f(A) = [M, S] = [-1, 4]. \quad \diamond$$

### Example 2.2

**A.** Find maximum and minimum of the  $C^\infty$ -function

$$f(x, y) = x^4 + 4x^2y^2 + y^4 - 4x^3 - 4y^3$$

in the set  $A$  given by  $x^2 + y^2 \leq 4 = 2^2$ .

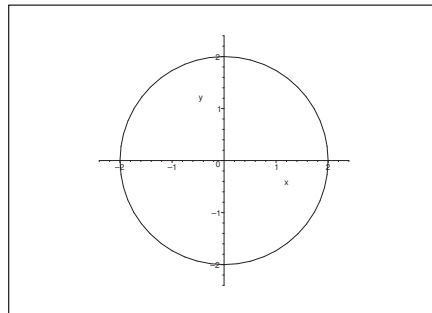


Figure 3: The domain  $A$ .

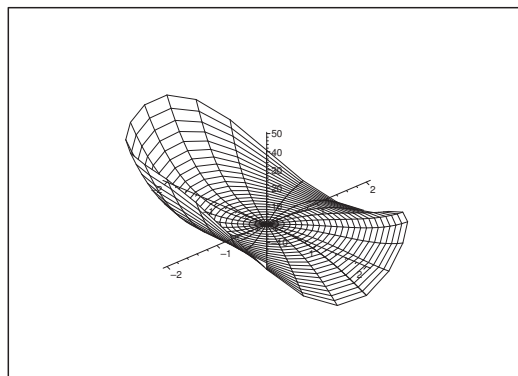


Figure 4: The graph of  $f(x, y)$  over  $A$ . Notice that a consideration of the graph does not give any hint.

D. Even if the rewriting of the function

$$f(x, y) = (x^2 + y^2)^2 + 2x^2y^2 - 4(x^3 + y^3)$$

looks reasonably nice it is still not tempting to apply an analysis of the level curves  $f(x, y) = c$ , so we shall again use the standard method as described in the previous example, to which we refer for the description.

- I. 1) The domain  $A$  has been sketched already. Since  $A$  is *closed and bounded*, and  $f(x, y)$  is *continuous* on  $A$ , it follows from the *second main theorem for continuous functions* that  $f(x, y)$  has a maximum and a minimum on  $A$ .

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While we are dealing with theoretical considerations we may aside mention that since  $A$  is obviously *connected*, it follows from the *first main theorem for continuous functions* that the range is connected, i.e. an interval, which necessarily is given by

$$f(A) = [M, S].$$

- 2) Since  $f(x, y)$  is of class  $C^\infty$ , there is *no exceptional point*.  
 3) The *stationary points* (if any) satisfies the system of equations

$$0 = \frac{\partial f}{\partial x} = 4x^3 + 8xy^2 - 12x^2 = 4x(x^2 + 2y^2 - 3x),$$

$$0 = \frac{\partial f}{\partial y} = 8x^2y + 4y^3 - 12y^2 = 4y(2x^2 + y^2 - 3y).$$

Note that it is *extremely important* to factorize the expressions as much as possible in order to solve the system. In fact, when this is done, we can reduce the system to

$$\frac{\partial f}{\partial x} = 0: \quad x = 0 \quad \text{or} \quad x^2 + 2y^2 - 3x = 0,$$

$$\frac{\partial f}{\partial y} = 0: \quad y = 0 \quad \text{or} \quad 2x^2 + y^2 - 3y = 0.$$

These conditions are now paired in  $2 \cdot 2 = 4$  ways which are handled one by one.

- a) When  $x = 0$  and  $y = 0$ , we get  $(0, 0) \in A^\circ$ , i.e.  $(0, 0)$  is a stationary point with the value of the function

$$f(0, 0) = 0.$$

- b) When  $x = 0$  and  $2x^2 + y^2 - 3y = 0$ , we get

$$0 + y^2 - 3y = y(y - 3) = 0, \quad \text{hence } y = 0 \text{ or } y = 3.$$

Thus, we have two possibilities:  $(0, 0) \in A^\circ$ , which has already been found previously, and  $(0, 3) \notin A$ , so this point does not participate in the competition. We therefore do not get further points in this case.

- c) When  $y = 0$  and  $x^2 + 2y^2 - 3x = 0$ , we get by an interchange of letters  $(x, y) \rightarrow (y, x)$  that the candidates are  $(0, 0) \in A^\circ$  [found previously] and  $(3, 0) \notin A$ . Hence we get no further point in this case.

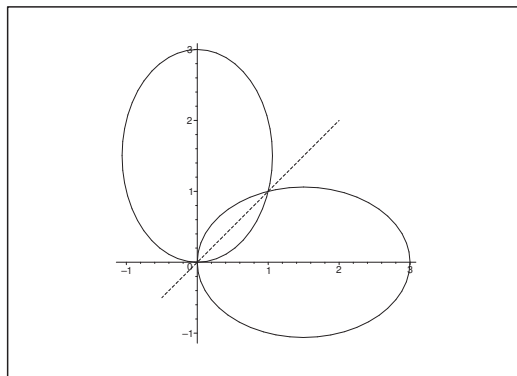


Figure 5: The ellipses  $x^2 + 2y^2 - 3x = 0$  and  $2x^2 + y^2 - 3y = 0$  and the line of symmetry  $y = x$ .

d) It still remains the last possibility

$$x^2 + 2y^2 - 3x = 0 \quad \text{and} \quad 2x^2 + y^2 - 3y = 0.$$

From the rewriting (cf. e.g. Linear Algebra)

$$\left(x - \frac{3}{2}\right)^2 + 2y^2 = \left(\frac{3}{2}\right)^2 \quad \text{and} \quad 2x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$

it is seen that the stationary points are the intersections of the two ellipses. It follows from the *symmetry* that the points must lie on the line  $y = x$ . By eliminating  $y$  we get

$$0 = x^2 + 2y^2 - 3x = 3x^2 - 3x = 3x(x - 1).$$

Hence we get either  $x = 0$ , corresponding to  $(0, 0) \in A^\circ$  [found previously] or  $x = 1$  corresponding to  $(1, 1) \in A^\circ$ , which is a new candidate with the value

$$f(1, 1) = 1 + 4 + 1 - 4 - 4 = -2.$$

Summarizing we get the *stationary points*  $(0, 0)$  and  $(1, 1)$  with the corresponding values of the function

$$f(0, 0) = 0 \quad \text{og} \quad f(1, 1) = -2.$$

4) The *boundary*. The simplest version is the following *alternative* to the standard procedure: A parametric representation of the boundary curve is

$$(x, y) = \mathbf{r}(\varphi) = (2 \cos \varphi, 2 \sin \varphi), \quad \varphi \in [0, 2\pi], \quad (\text{evt. } \varphi \in \mathbb{R}),$$

where we note that

$$(1) \quad \left(\frac{dx}{d\varphi}, \frac{dy}{d\varphi}\right) = \mathbf{r}'(\varphi) = (-2 \sin \varphi, 2 \cos \varphi) = (-y, x).$$

If we put  $g(\varphi) = f(\mathbf{r}(\varphi))$ , where

$$f(x, y) = x^4 + 4x^2y^2 + y^4 - 4x^3 - 4y^3,$$

then we get by the *chain rule*, that the maximum and the minimum on the boundary should be searched among the points on the *boundary*

$$x^2 + y^2 = 4,$$

for which (apply (1)),

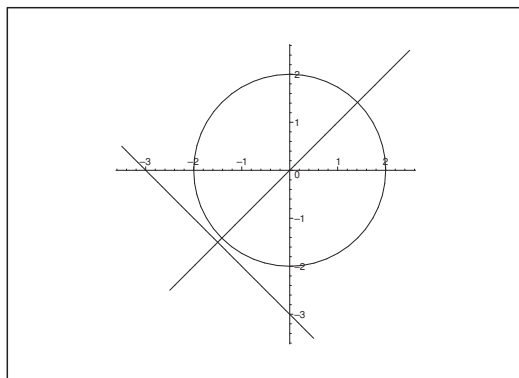


Figure 6: The intersections of the circle and the lines  $x = 0$ ,  $y = 0$ ,  $y = x$  and  $x + y + 3 = 0$ .

$$\begin{aligned}
 0 &= g'(\varphi) = \frac{\partial f}{\partial x} \cdot \frac{dx}{d\varphi} + \frac{\partial f}{\partial y} \cdot \frac{dy}{d\varphi} \\
 &= \{4x^3 + 8xy^2 - 12x^2\} \cdot (-y) + \{8x^2y + 4y^3 - 12y^2\} x \\
 &= 4x \{x^2 + 2y^2 - 3x\} (-y) + 4y \{2x^2 + y^2 - 3y\} x \\
 &= 4xy \{-x^2 - 2y^2 + 3x + 2x^2 + y^2 - 3y\} \\
 &= 4xy \{x^2 - y^2 + 3(x - y)\} \\
 &= 4xy(x - y)\{3 + x + y\}.
 \end{aligned}$$

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Hence we shall find the intersections between the *circle*  $x^2 + y^2 = 4 = 2^2$  and the *lines*

$$x = 0, \quad y = 0, \quad y = x \quad \text{and} \quad x + y + 3 = 0.$$

It follows immediately that these intersections are

$$(2, 0), \quad (\sqrt{2}, \sqrt{2}), \quad (0, 2), \quad (-2, 0), \quad (-\sqrt{2}, -\sqrt{2}), \quad (0, -2).$$

We take a note of the values

$$f(2, 0) = f(0, 2) = 16 - 32 = -16,$$

$$f(-2, 0) = f(0, -2) = 16 + 32 = 48,$$

$$f(\sqrt{2}, \sqrt{2}) = 6 \cdot 4 - 2 \cdot 4 \cdot 2\sqrt{2} = 24 - 16\sqrt{2},$$

$$f(-\sqrt{2}, -\sqrt{2}) = 24 + 16\sqrt{2}.$$

5) Summarizing we shall compare *numerically*

exceptional points: none,

stationary points:  $f(0, 0) = 0, \quad f(1, 1) = -2,$

boundary points:  $f(2, 0) = f(0, 2) = -16,$

$$f(-2, 0) = f(0, -2) = 48,$$

$$f(\sqrt{2}, \sqrt{2}) = 24 - 16\sqrt{2},$$

$$f(-\sqrt{2}, -\sqrt{2}) = 24 + 16\sqrt{2}.$$

Since  $16\sqrt{2} < 16 \cdot \frac{3}{2} = 24$ , it follows that

the minimum is  $M = f(2, 0) = f(0, 2) = -16,$

the maximum is  $S = f(-2, 0) = f(0, -2) = 48,$

and that both the minimum and the maximum are lying on the boundary.

6) Finally, we get from 1) that due to the first main theorem for continuous functions the range is the interval

$$f(A) = [M, S] = [-16, 48]. \quad \diamond$$

### Example 2.3

**A.** Find maximum and minimum for the function

$$f(x, y) = \sqrt{x^2 + 16y^2} - y^4$$

in the set

$$A = \{(x, y) \mid x^2 + 36y^2 \leq 81\}.$$

D. In this case one might find the level curves  $f(x, y) = c$ , which by using that

$$a^2 - b^2 = (a + b)(a - b)$$

can be rewritten as

$$x^2 = (y^4 + c)^2 - 16y^2 = (y^4 + 4y + c)(y^4 - 4y + c).$$

This expression still looks too difficult to analyze, so we shall again stick to the standard procedure as described in the first example.

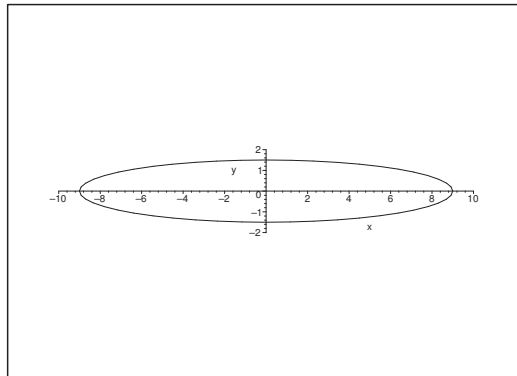


Figure 7: The closed and bounded domain  $A$ .

I. 1) Using some Linear Algebra, the set  $A$  is written as

$$\left(\frac{x}{9}\right)^2 + \left(\frac{y}{\frac{3}{2}}\right)^2 \leq 1,$$

which shows that  $A$  is a closed ellipsoidal disc, cf. the figure.

Since the set  $A$  is *closed and bounded*, and even *connected*, and  $f(x, y)$  is *continuous* on  $A$ , it follows from the *second main theorem for continuous functions* that  $f$  has a minimum  $M$  and a maximum  $S$  on  $A$ . It follows furthermore from the *first main theorem for continuous functions* that the range is connected, i.e. an interval, which necessarily is

$$f(A) = [M, S].$$

2) Since the square root is *not* differentiable at 0, it follows that  $(0, 0)$  is an *exceptional point!* We make a note for 5) of the value

$$f(0, 0) = 0.$$



3) The *stationary points* in  $A^\circ \setminus \{(0,0)\}$ , if any, must satisfy the system of equations

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + 16y^2}} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{16y}{\sqrt{x^2 + 16y^2}} - 4y^3 = 0.$$

The first equation is only fulfilled for  $x = 0$ . Thus any stationary point must lie on the  $y$ -axis.

Since  $(0,0)$  is an exceptional point, we must have  $y \neq 0$  for any stationary point. When we put  $x = 0$  into the second equation, we get (NB:  $\sqrt{y^2} = |y|$ )

$$0 = \frac{16y}{\sqrt{16y^2}} - 4y^3 = 4y \left\{ \frac{1}{|y|} - y^2 \right\} = 4 \frac{y}{|y|} \{1 - |y|^3\}.$$

Since  $y \neq 0$ , we must have  $|y| = 1$ , i.e.  $y = \pm 1$ . Hence the *stationary points* are  $(0,1)$  and  $(0,-1)$ . We make a note for 5) of the value

$$f(0,1) = f(0,-1) = \sqrt{16} - 1 = 3.$$

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4) The *boundary*. On the boundary we get  $x^2 + 36y^2 = 81$ , i.e.

$$x^2 = 81 - 36y^2.$$

Since  $f(x, y)$  *only* contains  $x$  in the form  $x^2$ , we can use this equation to eliminate  $x^2$  when we write down the restriction,

$$\begin{aligned} f(y) &= \sqrt{x^2 + 16y^2} - y^4 = \sqrt{81 - 36y^2 + 16y^2} - y^4 \\ &= \sqrt{81 - 20y^2} - y^4 \quad y \in \left[-\frac{3}{2}, \frac{3}{2}\right]. \end{aligned}$$

It follows immediately that  $g(y)$  is *decreasing* in the new variable  $t = y^2 \in \left[0, \frac{9}{4}\right]$ , hence the maximum on the boundary is

$$g(0) = f(-9, 0) = f(9, 0) = 9,$$

and the minimum on the boundary is

$$g\left(\pm\frac{3}{2}\right) = f\left(0, \frac{3}{2}\right) = f\left(0, -\frac{3}{2}\right) = \sqrt{16 \cdot \frac{9}{4}} - \frac{81}{16} = 6 - \frac{81}{16} = \frac{15}{16}.$$

5) A *numerical comparison* of

$$\text{exceptional point: } f(0, 0) = 0,$$

$$\text{stationary points: } f(0, 1) = f(0, -1) = 3,$$

$$\text{boundary points: } f\left(0, \frac{3}{2}\right) = f\left(0, -\frac{3}{2}\right) = \frac{15}{16},$$

$$f(-9, 0) = f(9, 0) = 9,$$

gives

$$\text{maximum: } f(-9, 0) = f(9, 0) = 9, \quad (\text{boundary points}),$$

$$\text{minimum: } f(0, 0) = 0, \quad (\text{exceptional point}).$$

6) According to 1) the range is given by

$$f(A) = [M, S] = [0, 9],$$

where we have used the *first main theorem for continuous functions*.  $\diamond$

**Example 2.4**

A. Consider the function

$$f(x, y) = x + 3y - 2 \ln(1 + 4xy)$$

defined on the triangle  $A$  with its vertices  $(1, 0)$ ,  $(4, 0)$  and  $(1, 1)$ . Find the maximum and minimum of  $f(x, y)$  on  $A$ .

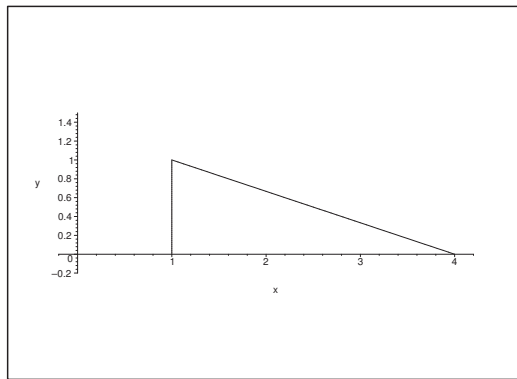


Figure 8: The closed and bounded domain  $A$ .

D. Here it is totally out of question to find the level curves, so we apply the standard procedure as described in Example 2.1.

I. 1) We first sketch  $A$ . Since  $f(x, y)$  is *continuous* on the *closed and bounded* triangle  $A$  (note in particular that  $1 + 4xy > 0$ ), it follows from the *second main theorem for continuous functions* that  $f(x, y)$  has both a maximum  $S$  and a minimum  $M$  on  $A$ . Since  $A$  is also connected, it follows from the *first main theorem for continuous functions* that the range is connected, i.e. an interval, and we have necessarily

$$f(A) = [M, S].$$

2) Since  $f$  everywhere in  $A^\circ$  is of class  $C^\infty$ , it follows that  $f(x, y)$  has *no exceptional point*.

3) The *stationary points*, if any, must satisfy the equations

$$\frac{\partial f}{\partial x} = 1 - \frac{8y}{1 + 4xy} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3 - \frac{8x}{1 + 4xy} = 0,$$

i.e.

$$8y = 1 + 4xy \quad \text{and} \quad 8x = 3(1 + 4xy).$$

When  $1 + 4xy > 0$  is eliminated we get  $8x = 3 \cdot 8y$ , from which  $x = 3y$ , which is a condition that the stationary points necessarily must satisfy.

By insertion of  $x = 3y$  we get

$$8y = 1 + 4xy = 1 + 12y^2,$$

which is rewritten as

$$0 = 12y^2 - 8y + 1 = 12 \left( y - \frac{1}{6} \right) \left( y - \frac{1}{2} \right).$$

From this we either get  $y = \frac{1}{6}$ , corresponding to  $x = 3 \cdot \frac{1}{6} = \frac{1}{2}$ , i.e.  $\left( \frac{1}{2}, \frac{1}{6} \right) \notin A$ , or  $y = \frac{1}{2}$ , corresponding to

$$\left( \frac{3}{2}, \frac{1}{2} \right) \in A^\circ.$$

We only find one stationary point  $\left( \frac{3}{2}, \frac{1}{2} \right)$ . We make a note of the value for 5) below,

$$f \left( \frac{3}{2}, \frac{1}{2} \right) = \frac{3}{2} + \frac{3}{2} - 2 \ln \left( 1 + 4 \cdot \frac{3}{2} \cdot \frac{1}{2} \right) = 3 - 2 \ln 4 = 3 - 4 \ln 2.$$

4) The *investigation of the boundary* is divided into three cases:

a) On the line  $x = 1$ ,  $y \in [0, 1]$ , we get the restriction

$$g_1(y) = 1 + 3y - 2 \ln(1 + 4y),$$

where

$$g'_1(y) = 3 - \frac{8}{1 + 4y} = 0 \quad \text{for } 1 + 4y = \frac{8}{3}, \text{ i.e. } y = \frac{5}{12} \in [0, 1],$$

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corresponding to

$$f\left(1, \frac{5}{12}\right) = g_1\left(\frac{5}{12}\right) = 1 + \frac{5}{4} - 2\ln\left(1 + \frac{5}{3}\right) = \frac{9}{4} - 2\ln\left(\frac{8}{3}\right).$$

NB: We must *not* forget the endpoints of the line:

$$f(1, 0) = g_1(0) = 1 + 0 - 2\ln(1 + 4 \cdot 0) = 1,$$

$$f(1, 1) = g_1(1) = 1 + 3 - 2\ln(1 + 4 \cdot 1) = 4 - 2\ln 5.$$

b) On the line  $y = 0$ ,  $x \in [1, 4]$ , we get the restriction

$$g_2(x) = x - 2\ln(1 + 4 \cdot x \cdot 0) = x,$$

which obviously is increasing. Therefore we shall only make a note on the values at the endpoints,

$$f(1, 0) = 1 \quad \text{and} \quad f(4, 0) = 4.$$

c) On the line  $x + 3y = 4$ , i.e.  $x = 4 - 3y$ ,  $y \in [0, 1]$ , the restriction is given by

$$g_3(y) = 4 - 2\ln(1 + 4(4 - 3y)y) = 4 - 2\ln(1 + 16y - 12y^2).$$

Here we get

$$g_3'(y) = -\frac{2}{1+16y-12y^2} (16-24y) = 0 \quad \text{for } y = \frac{2}{3} \in [0, 1],$$

corresponding to  $x = 4 - 3 \cdot \frac{2}{3} = 2$ . The interesting point is  $\left(2, \frac{2}{3}\right) \in \partial A$  with the value

$$\begin{aligned} f\left(2, \frac{2}{3}\right) &= g_3\left(\frac{2}{3}\right) = 4 - 2\ln\left(1 + 16 \cdot \frac{2}{3} - 12 \cdot \frac{4}{9}\right) \\ &= 4 - 2\ln\left(1 + \frac{32}{3} - \frac{16}{3}\right) = 4 - 2\ln \frac{19}{3}. \end{aligned}$$

We have already treated the two endpoints earlier.

5) Finally we shall compare *numerically*

exceptional points: none,

$$\text{stationary point: } f\left(\frac{3}{2}, \frac{1}{2}\right) = 3 - 4\ln 2 \approx 0, 23,$$

$$\text{boundary a): } f\left(1, \frac{5}{12}\right) = \frac{9}{4} - 2\ln\left(\frac{8}{3}\right) \approx 0, 29,$$

$$f(1, 1) = 4 - 2\ln 5 \approx 0, 79,$$

$$f(1, 0) = 1,$$

$$\text{boundary b): } f(4, 0) = 4,$$

$$\text{boundary c): } f\left(2, \frac{2}{3}\right) = 4 - 2\ln \frac{19}{3} \approx 0, 31.$$

By a comparison we see that

the maximum is  $S = f(0, 4) = 4$ , (boundary point),

the minimum is  $M = f\left(\frac{3}{2}, \frac{1}{2}\right) = 3 - 4 \ln 2$ , (stationary point).

**Remark 2.3** Note that the comparison is made approximatively, while the result is given in an exact form.  $\diamond$

6) According to 1) we finally get by the *first main theorem for continuous functions* that the range is

$$f(A) = [M, S] = [3 - 4 \ln 2, 4]. \quad \diamond$$

**Example 2.5** A nasty example which usually is not given in any textbook, is given by the following. It illustrates that the usual division of cases in the textbooks is not exhaustive.

Let  $A = K(\mathbf{0}; 1)$  be the open unit disc, and consider the function

$$f(x, y) = (x^2 + y^2) \cos\left(\frac{1}{1 - x^2 - y^2}\right), \quad x^2 + y^2 < 1.$$

Then  $f(x, y)$  is bounded on  $A$ ,

$$|f(x, y)| \leq x^2 + y^2 < 1 \quad \text{for } (x, y) \in A,$$

and we see that  $f(x, y)$  has no continuous extension to any point on the boundary.

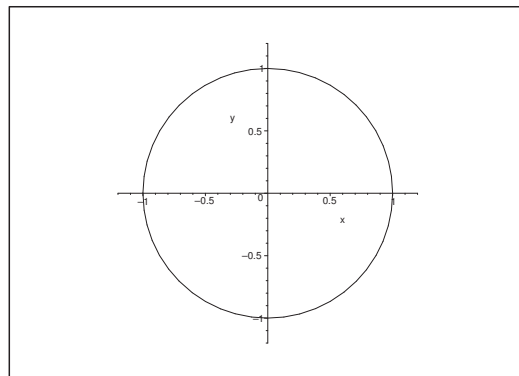


Figure 9: The set  $A$  is the open unit disc.

Then note that

$$1) f(x, y) = 1 - \frac{1}{2p\pi} \text{ for } x^2 + y^2 = 1 - \frac{1}{2p\pi}, p \in \mathbb{N},$$

$$2) f(x, y) = -1 + \frac{1}{(2p+1)\pi} \text{ for } x^2 + y^2 = 1 - \frac{1}{(2p+1)\pi}, p \in \mathbb{N},$$

from which we conclude that  $f$  has neither a maximum nor a minimum in the open set  $A$ .

However, since  $f(x, y)$  is *continuous* on the *connected* set  $A$ , it follows from the *first main theorem for continuous functions* that  $f(A)$  also is connected, i.e. an interval.

According to 1) the function  $f(x, y)$  attains values smaller than 1, though we can get as close to 1 as we wish.

According to 2) the function  $f(x, y)$  attains values bigger than  $-1$ , though we can get as close to  $-1$  as we wish.

Hence we conclude that the range is given by

$$f(A) = ] - 1, 1[. \quad \diamond$$

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**Example 2.6**

Let  $A$  be the open triangle

$$A = \{(x, y) \mid 0 < x < 1, -x < y < 4x\},$$

and let the function  $f(x, y)$  on  $A$  be given by

$$f(x, y) = 2xy + 3 \ln(1 - x), \quad (x, y) \in A.$$

Find the range  $f(A)$ .

**A.**

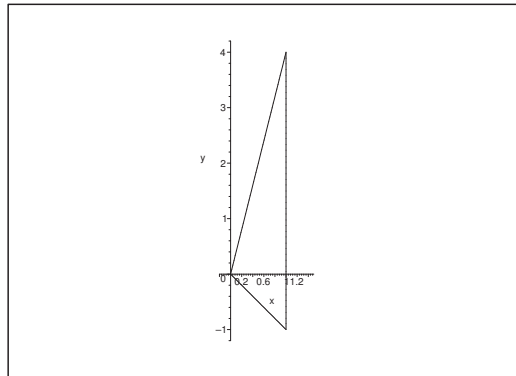


Figure 10: The open and bounded domain  $A$ .

**D.** Here it is possible to find the *level curves*. In fact, since  $x > 0$  in  $A$ , we get that

$$f(x, y) = 2xy + 3 \ln(1 - x) = c, \quad (x, y) \in A,$$

is equivalent to

$$y = \varphi_c(x) = \frac{c}{2x} - \frac{3}{2} \cdot \frac{\ln(1 - x)}{x}.$$

Although the expression looks very complicated, it is actually possible to analyze these level curves. The reader is referred to section **I 2** which, however, may be considered a bit advanced for a common use.

We therefore start with the *standard procedure* in section **I 1** with some necessary modifications. First we exploit the theoretical main theorems as much as possible. Then we extend  $f$  to the parts of the boundary where it is possible, and we discuss what happens at the boundary points where such a continuous extension of  $f$  is not possible.

We see that both methods have a common theoretical start, which we here call section **I**.



- I. Since  $f(x, y)$  is *continuous* on the *connected* set  $A$ , it follows from the *first main theorem for continuous functions* that the range  $f(A)$  is *connected*, i.e. an interval.

Since  $A$  is bounded, though *not closed*, we cannot apply the *second main theorem for continuous functions*. We shall first find out whether  $f(x, y)$  has a continuous extension to (parts of) the boundary of  $A$ .

It follows immediately that  $f(x, y)$  can be continuously extended to the lines

$$y = 4x \quad \text{and} \quad y = -x, \quad x \in [0, 1[,$$

with the same formal expression of the function, i.e. the extension is given by

$$\bar{f}(x, y) = 2xy + 3 \ln(1 - x) \quad \text{for } 0 \leq x < 1, \quad -x \leq y \leq 4x.$$

On the other hand, we cannot extend to the vertical line  $x = 1$ , because

$$\lim_{x \rightarrow 1^-} f(x, y) = 2y + 3 \lim_{x \rightarrow 1^-} \ln(1 - x) = -\infty.$$

However, we see that the *lower bound* is  $-\infty$ , so  $f(A)$  must be a semi-infinite, i.e. either  $] - \infty, a[$  or  $] - \infty, a]$ , because the theorems do not assure that the upper bound  $a$  actually belongs to  $f(A)$ . This question can only be decided by an explicit analysis.

It follows that we shall only search the *maximum* in

$$B = \{(x, y) \mid 0 \leq x < 1, -x \leq y \leq 4x\}.$$

Since we also have  $f(x, y) \rightarrow -\infty$  for  $x \rightarrow 1^-$ , in  $B$ , there exists an  $\varepsilon \in ]0, 1[$ , such that

$$\bar{f}(x, y) < S \quad \text{for } (x, y) \in \bar{B} \text{ and } 1 - \varepsilon \leq x < 1.$$

The maximum  $S$  is therefore attained in the *closed and bounded* and truncated domain

$$B_\varepsilon = \{(x, y) \mid 0 \leq x \leq 1 - \varepsilon, -x \leq y \leq 4x\},$$

where we of course assume that  $S$  exists and  $S < +\infty$ .

This follows, however, from the *second main theorem for continuous functions*, applied on  $B_\varepsilon$ .

Since we only want to find the maximum, the standard procedure is hereafter the same as for closed and bounded domains. The only modification is that we shall not go through an investigation of the boundary on the line  $x = 1 - \varepsilon$ .

#### I1. Standard procedure.

- 1) We have already sketched a figure and quoted and applied the second main theorem.
- 2) Since  $f(x, y)$  belongs to the class  $C^\infty$  in  $A$ , there is *no exceptional point*.
- 3) The *stationary points* in  $A$ , if any, must satisfy the equations

$$\frac{\partial f}{\partial x} = 2y - \frac{3}{1-x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2x = 0.$$

It follows from the latter equation that  $x = 0$ ; but since  $x > 0$  in  $A$ , we see that we have *no stationary point* in  $A$  for the function  $f$ .

4) *Modified investigation of the boundary.*a) For  $y = 4x$  we get the restriction

$$g_1(x) = 8x^2 + 3 \ln(1 - x), \quad \text{for } x \in [0, 1[,$$

where

$$g_1'(x) = 16x - \frac{3}{1-x}.$$

Hence,  $g_1'(x) = 0$  for

$$0 = 16x^2 - 16x + 3 = (4x - 3)(4x - 1),$$

i.e. for  $x = \frac{1}{4}$  or  $x = \frac{3}{4}$ .

By applying high school calculus it is seen that the maximum is either attained for  $x = 0$ , corresponding to  $g_1(0) = \bar{f}(0, 0) =$ , or for  $x = \frac{3}{4}$ , corresponding to

$$\begin{aligned} g_1\left(\frac{3}{4}\right) &= \bar{f}\left(\frac{3}{4}, 3\right) = \frac{8 \cdot 9}{16} + 3 \ln\left(1 - \frac{3}{4}\right) = \frac{9}{2} - 6 \ln 2 \\ &\geq 4,5 - 6 \cdot 0,7 = 0,3 > 0. \end{aligned}$$

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b) For  $y = -x$  we get the restriction

$$g_2(x) = -2x^2 + 3 \ln(1 - x), \quad \text{for } x \in [0, 1[,$$

where

$$g_2'(x) = -4x - \frac{3}{1-x} < 0.$$

Hence,  $g_2(x)$  is *decreasing*. The maximum on this line is therefore  $g_2(0, 0) = \bar{f}(0, 0) = 0$ .

c) *Numerical comparison*. When we compare the values of the candidates above it follows that the maximum in  $B$  is

$$\bar{f}\left(\frac{3}{4}, 3\right) = \frac{9}{2} - 6 \ln 2 > 0.$$

This value is *only* attained at the boundary point  $\left(\frac{3}{4}, 3\right)$ , so

$$\bullet \bar{f}(B) = \left] -\infty, \frac{9}{2} - 6 \ln 2 \right],$$

and

$$\bullet f(A) = \left] -\infty, \frac{9}{2} - 6 \ln 2 \right[,$$

because  $A$  is obtained by removing all boundary points from  $B$ .

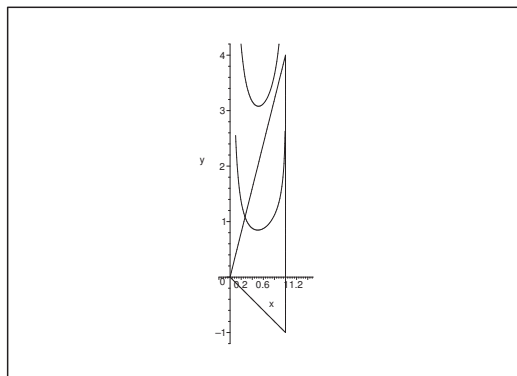


Figure 11: The level curves for  $c = \frac{1}{2}$  (below) and  $c = 1$  (above).

**I2.** *The method of level curves.* The level curve

$$y = \varphi_c(x) = \frac{c}{2x} - \frac{3}{2} \cdot \frac{\ln(1-x)}{x}$$

is defined in the strip  $0 < x < 1$  as the graph of a function. If  $c \neq 0$ , then both  $x = 0$  and  $x = 1$  are asymptotes. It follows that

$$\lim_{x \rightarrow 1^-} \varphi_c(x) = +\infty \quad \text{for all } c \in \mathbb{R},$$

and that

$$\lim_{x \rightarrow 0^+} \varphi_c(x) = +\infty \quad \text{for } c > 0,$$

and

$$\lim_{x \rightarrow 0^+} \varphi_c(x) = -\infty \quad \text{for } c < 0.$$

The curves are characterized by  $f(x, y)$  being constant  $c$  along  $y = \varphi_c(x)$ . We have sketched two level curves on the figure (where  $c > 0$ ), from which it is seen that the curved “move upwards”, when  $c$  increases.

Hence we are looking for the biggest  $c$ , for which  $y = \varphi_c(x)$  just is contacting the boundary of  $B$  without intersecting  $B$ . This is not possible for  $c < 0$ , and at the same time we get the line  $y = -x$  excluded. Thus the maximum can *only* lie on the line  $y = 4x$ . Since  $y = \varphi_c(x)$  only *touches* this line, the following two conditions must be fulfilled:

- 1) The curves must go through the *same point*, i.e.  $y = 4x = \varphi_c(x)$ , or

$$4x = -\frac{1}{2x} \{-c + 3 \ln(1 - x)\},$$

from which

$$-c + 3 \ln(1 - x) = -8x^2.$$

- 2) The curves must have the *same slope* at this point, i.e.

$$4 = \varphi'_c(x) = \frac{1}{2x^2} \left\{ -c + 3 \ln(1 - x) + \frac{3x}{1 - x} \right\}.$$

The ugly terms  $-c + 3 \ln(1 - x)$  in 2) can be eliminated by applying 1), hence

$$8x^2 = -8x^2 + \frac{3x}{1 - x},$$

which is rewritten as

$$0 = 16x^2(1 - x) - 3x = x\{16x - 16x^2 - 3\} = -x(4x - 1)(4x - 3).$$

From this we get the solutions  $x = 0$ ,  $x = \frac{1}{4}$  and  $x = \frac{3}{4}$ , and since  $y = 4x$ , we finally get the *candidates*

$$(0, 0), \quad \left(\frac{1}{4}, 1\right), \quad \left(\frac{3}{4}, 3\right),$$

with the corresponding function values for the extended function,

$$\bar{f}(0, 0) = 0, \quad \bar{f}\left(\frac{1}{4}, 1\right) = \frac{1}{2} - 3 \ln \frac{4}{3}, \quad \bar{f}\left(\frac{3}{4}, 3\right) = \frac{9}{2} - 6 \ln 2.$$

By a numerical comparison we get that the maximum is attained at the point  $\left(\frac{3}{4}, 3\right)$ . Hence we conclude that

$$\bar{f}(B) = -\left] -\infty, \frac{9}{2} - 6 \ln 2 \right].$$

Finally, when we remove the boundary points from  $B$ , we obtain as previously that

$$f(A) = -\left] -\infty, \frac{9}{2} - 6 \ln 2 \right[. \quad \diamond$$

## 2.2 Extremum

**Example 2.7** In this example we produce some functions in  $\mathbb{R}^2$ , which all have  $(0, 0)$  as a stationary point and value zero.

We supply the investigation with sketches of the graphs and discussions of the sign of the function in the neighbourhood whenever this is necessary. Concerning the graphs the reader is also referred to Linear Algebra.

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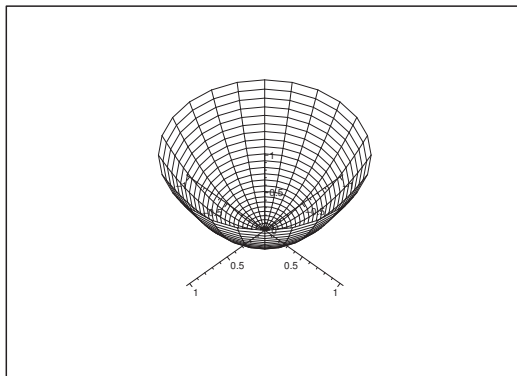


Figure 12: The graph of  $z = x^2 + y^2$ .

1)  $z = f_1(x, y) = x^2 + y^2$ .

The graph of  $f_1$  is a *paraboloid of revolution*.

Since

$$f_1(x, y) > 0 = f(0, 0) \quad \text{for } (x, y) \neq (0, 0),$$

the function  $f_1$  has a *true minimum* at  $(0, 0)$ . It is easily seen that this is also the *global minimum* of the function.

2)  $z = f_2(x, y) = x^2 - 4xy + 4y^2 = (x - 2y)^2$ .

The graph of  $f_2$  is a *parabolic cylinder*. It follows immediately that

$$f_2(x, y) \geq 0 = f(0, 0);$$

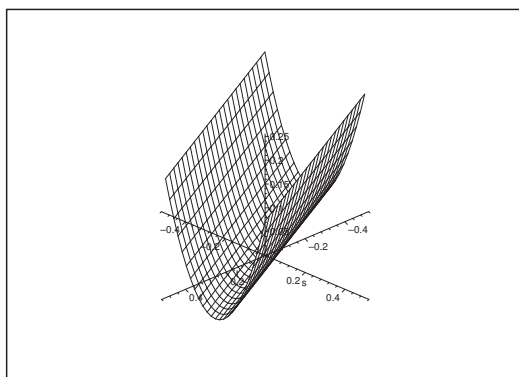


Figure 13: The graph of  $z = (x - 2y)^2$ .

but since

$$f\left(x, \frac{x}{2}\right) = 0 = f(0, 0) \quad \text{for alle } x,$$

we see that  $(0, 0)$  is a *weak local minimum*. However, we also have in this case that 0 is a *global minimum*.

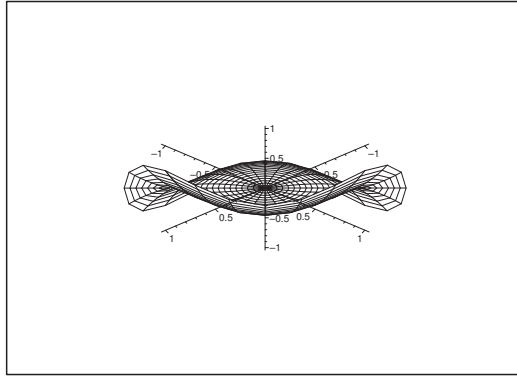


Figure 14: The graph of  $z = x^3 + y^3$  sketched in MAPLE does not give the best picture.

$$3) \quad z = f_3(x, y) = x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

Since  $x^3 + y^3$  is of *odd degree 3*, we take e.g. the restriction of  $f_3$  to the  $x$ -axis,

$$f_3(x, 0) = x^3 \quad (\text{is both } > \text{ and } < 0),$$

in a neighbourhood of  $x = 0$ , so  $f_3$  has no extremum at  $(0, 0)$ .

This can also be seen by analyzing the sign of the function. In fact,  $x^2 - xy + y^2 \geq 0$  for all  $(x, y)$ , thus  $x^3 + y^3$  is everywhere of the same sign as  $x + y$ .

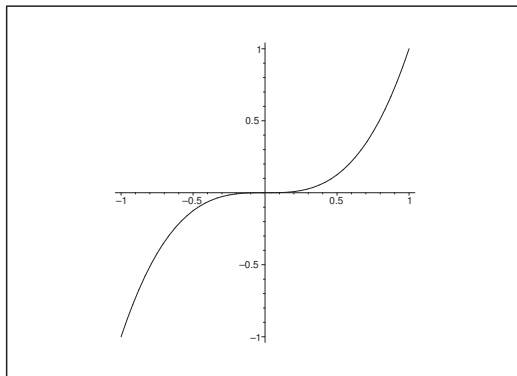
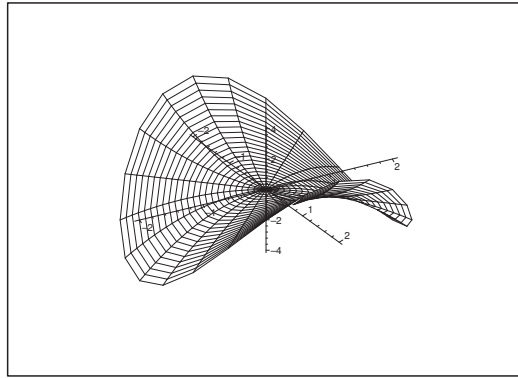


Figure 15: The restriction to the  $x$ -axis gives a better picture.

Figure 16: The graph of  $z = x^2 - y^2$ .

$$4) z = f_4(x, y) = x^2 - y^2 = (x + y)(x - y).$$

The graph is a *hyperbolic paraboloid*. There is *no* extremum at  $(0, 0)$ .

An *analysis of the sign* shows that  $f_4(x, y)$  is 0 on the lines  $x + y = 0$  and  $x - y = 0$ , and that  $f_4(x, y)$  attains both positive and negative values in any neighbourhood of  $(0, 0)$ .

It is finally also possible to consider the restrictions

$$\begin{aligned} x\text{-axis: } f_4(x, 0) &= x^2 > 0 && \text{for } x \neq 0, \\ y\text{-axis: } f_4(0, y) &= -y^2 < 0 && \text{for } y \neq 0, \end{aligned}$$

from which we arrive to the same conclusion.  $\diamond$

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**Example 2.8**

A. Examine whether the function

$$f(x, y, z) = \exp(xy + z^2)$$

has local extrema.

D. Here we shall not use the standard procedure but instead we suggest an *alternative* method. In fact, since  $\exp$  is *strictly increasing*, the functions

$$\varphi(x, y, z) = xy + z^2 \quad \text{and} \quad f(x, y, z) = \exp(\varphi(x, y, z))$$

must have the same stationary points and extrema.

We therefore examine the simpler function  $\varphi(x, y, z)$ .

I. The equations for the stationary points for  $\varphi(x, y, z)$  are

$$\frac{\partial \varphi}{\partial x} = y = 0, \quad \frac{\partial \varphi}{\partial y} = x = 0, \quad \frac{\partial \varphi}{\partial z} = 2z.$$

Hence it follows immediately that  $(0, 0, 0)$  is the only *stationary point*.

2) For the other candidates the  $(r, s, t)$ -method is easier, because it is the essence of the determination of the approximative polynomial of *at most degree two*. One often forgets in the applications that this is the general idea behind the  $(r, s, t)$ -method. Note that we in 1) had to expand to the third degree, which is the reason why the  $(r, s, t)$ -method fails for  $(0, 0)$ .

There is, however, also an *alternative* method for the other points. This will here be illustrated on the point  $(1, 1)$ .

a) First we *reset*, the problem, i.e. put  $(x, y) = (1 + h, 1 + k)$ , so  $(h, k) = (0, 0)$  corresponds to the point  $(x, y) = (1, 1)$  under examination.

b) Insert this in the expression for  $f(x, y)$  and write dots for terms of degree  $> 2$ :

$$\begin{aligned} f(x, y) &= (1 + h)^4 + 4(1 + h)^2(1 + k)^2 + (1 + k)^4 - 4(1 + h)^3 - 4(1 + k)^3 \\ &= 1 + 4h + 6h^2 + \dots + 4(1 + 2h + h^2)(1 + 2k + k^2) \\ &\quad + 1 + 4k + 6k^2 + \dots - 4(1 + 3h + 3h^2 + \dots) \\ &\quad - 4(1 + 3k + 3k^2 + \dots) \\ &= (1 + 4h + 6h^2) + 4 \{1 + 2h + h^2 + 2k + 4hk + k^2 + \dots\} \\ &\quad + (1 + 4k + 6k^2) - 4(1 + 3h + 3h^2) - 4(1 + 3k + 3k^2) + \dots \\ &= -2 - 2h^2 + 16hk - 2k^2 + \dots, \end{aligned}$$

i.e.

$$P_2(h, k) = -2 - 2(h^2 - 8hk + k^2) = -2 - 2 \{ (h - 4k)^2 - 15k^2 \}.$$

Since  $P_2(h, k) + 2$  attains both negative values (for  $k = 0$  and  $h \neq 0$ ) and positive values (for  $h = 4k$ ) in any neighbourhood of  $(h, k) = (0, 0)$ , we conclude that  $(x, y) = (1, 1)$  is *not* an extremum.  $\diamond$

**Example 2.10**

A. Examine whether the function

$$f(x, y) = 1 - 4x^2 - 4y^2 + x^2y^2, \quad (x, y) \in \mathbb{R}^2.$$

has any extremum. Find the range  $f(\mathbb{R}^2)$ .

D. When we apply the standard procedure we are guided through the usual examination of the exceptional points (there are none) and of the stationary points. It is, however, here possible to make a shortcut by noticing that  $f(x, y)$  only is a function in  $u = x^2$  and  $v = y^2$ , i.e.

$$f(x, y) = 1 - 4x^2 - 4y^2 + x^2y^2 = g(u, v) = 1 - 4u - 4v + uv, \quad u, v \geq 0.$$

I. The *stationary points* for  $f(x, y)$ , if any, must fulfil the equations

$$\frac{\partial f}{\partial x} = -2x(4 - y^2) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = -2y(4 - x^2) = 0.$$

This system is split into

$$x = 0 \quad \text{or} \quad y = 2 \quad \text{or} \quad y = -2,$$

and

$$y = 0 \quad \text{or} \quad x = 2 \quad \text{or} \quad x = -2.$$

Formally we get  $3 \cdot 3$  possibilities, but four of them are not possible (e.g.  $x$  cannot at the same time be 0 and 2 or  $-2$ ). We therefore get five *stationary points*,

$$(0, 0), \quad (2, 2), \quad (-2, -2), \quad (-2, 2), \quad (2, -2).$$

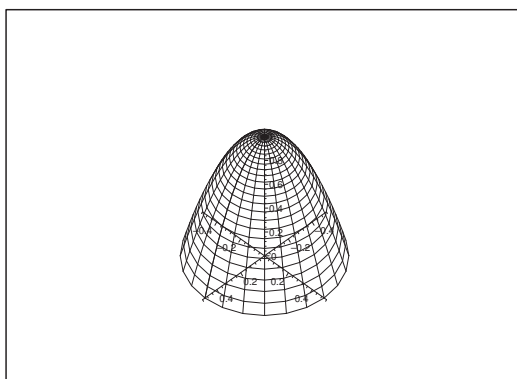


Figure 17: The graph of  $z = 1 - 4x^2 - 4y^2$ .

1) In  $(0, 0)$  we get

$$f(x, y) \approx P_2(x, y) = 1 - 4x^2 - 4y^2.$$

The graph of  $z = P_2(x, y)$  is an *elliptic paraboloid of revolution*. Obviously we have a *proper local maximum* at  $(0, 0)$ .

2) We take a shortcut by considering

$$g(u, v) = 1 - 4u - 4v + uv, \quad u = x^2, \quad v = y^2,$$

instead. First, all four stationary points for  $f$  are seen to correspond to the only point  $(u, v) = (4, 4)$ . It is therefore sufficient to examine  $g(u, v)$  in the neighbourhood of  $(4, 4)$ .

The approximating polynomial for  $g(u, v)$  expanded from  $(4, 4)$  of at most degree 2 is found by using:

$$g(u, v) = 1 - 4u - 4v + uv, \quad g(4, 4) = -15,$$

$$\frac{\partial g}{\partial u} = -4 + v, \quad g'_u(4, 4) = 0,$$

$$\frac{\partial g}{\partial v} = -4 + u, \quad g'_v(4, 4) = 0,$$

$$\frac{\partial^2 g}{\partial u^2} = 0,$$

$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial^2 g}{\partial v^2} = 0,$$

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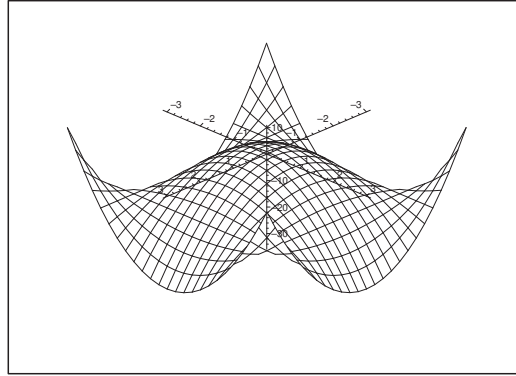


Figure 18: The graph of  $f(x, y) = 1 - 4x^2 - 4y^2 + x^2 + y^2$ .

hence

$$P_3(u, v) = -15 + \frac{1}{2} \cdot (u - 4)(v - 4) = -15 + (u - 4)(v - 4).$$

It follows that  $P_2(u, v)$  in the neighbourhood of  $(4, 4)$  attains values which are both  $> -15$  and  $< -15$ . Thus  $g(u, v)$  does *not* have an extremum at  $(4, 4)$ . This implies that  $f(x, y)$  does *not* have an extremum at  $(\pm 2, \pm 2)$  (all four possible combinations of the sign).

3) The function  $f(x, y)$  has only one local maximum,

$$f(0, 0) = 1.$$

However, this value is *not* the global maximum. In fact, by rewriting

$$f(x, y) = 1 - 4x^2 - 4y^2 + x^2 + y^2 = (x^2 - 4)(y^2 - 4) - 15,$$

we see that the restriction to the line  $y = x$  gives

$$f(x, x) = (x^2 - 4)^2 - 15 \rightarrow +\infty \quad \text{for } x \rightarrow \pm\infty.$$

4) Note also that

$$f(x, 0) = 1 - 4x^2 \rightarrow -\infty \quad \text{for } x \rightarrow \pm\infty.$$

Thus, since  $f$  is *continuous* on the *connected* set  $\mathbb{R}^2$ , it follows from the *first main theorem for continuous functions* that the range is

$$f(\mathbb{R}^2) = \mathbb{R}. \quad \diamond$$

### 3 The plane integral

#### 3.1 Rectangular coordinates

##### Example 3.1

A. Calculate  $\int_B xy \, dS$ , where  $B$  is given on the figure.

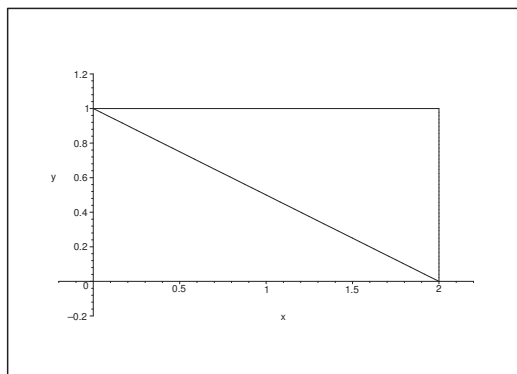


Figure 19: The domain  $B$  is the upper triangle.

D. We have two possibilities for the reduction:

D1. We first integrate horizontally.

D2. We first integrate vertically.

We shall treat both possibilities so we can compare the calculations in the two cases.

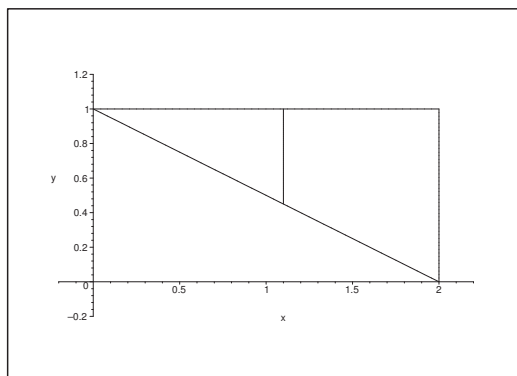


Figure 20: The domain  $B$  with the vertical integration line from  $y = 1 - \frac{1}{2}x$  to  $y = 1$ .

D1. We first integrate vertically.

I1. In this case we write the domain in the form

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 1 - \frac{1}{2}x \leq y \leq 1\}.$$

Notice that the *outer* variable  $x$  must always lie between two constants,  $0 \leq x \leq 2$ . Then we use the figure for any fixed  $x$  to find the integration interval for the *inner* variable of integration  $y$ , i.e. in this particular case  $1 - \frac{1}{2}x \leq y \leq 1$ .

Then write down the double integral:

$$(2) \int_B xy \, dS = \int_0^2 \left\{ \int_{1-\frac{1}{2}x}^1 xy \, dy \right\} dx = \int_0^2 x \left\{ \int_{1-\frac{1}{2}x}^1 y \, dy \right\} dx.$$

Calculate the inner integral,

$$\int_{1-\frac{1}{2}x}^1 y \, dy = \left[ \frac{1}{2} y^2 \right]_{1-\frac{1}{2}x}^1 = \frac{1}{2} \left\{ 1 - \left( 1 - \frac{1}{2}x \right)^2 \right\} = \frac{1}{2} \left\{ 1 - \frac{1}{4}x^2 \right\} = \frac{1}{2}x - \frac{1}{8}x^2.$$

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By insertion in (2), we get

$$\begin{aligned} \int_B xy \, dS &= \int_0^2 x \left\{ \frac{1}{2}x - \frac{1}{8}x^2 \right\} dx = \int_0^2 \left\{ \frac{1}{2}x^2 - \frac{1}{8}x^3 \right\} dx \\ &= \left[ \frac{1}{6}x^3 - \frac{1}{32}x^4 \right]_0^2 = \frac{8}{6} - \frac{1}{2} = \frac{5}{6}. \end{aligned}$$

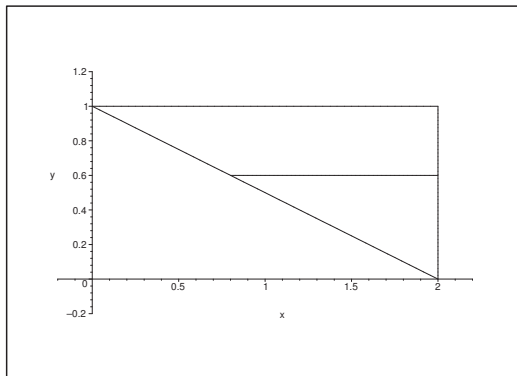


Figure 21: The domain  $B$  with the horizontal line of integration from  $x = 2 - 2y$  to  $x = 2$ .

**D 2.** Here we first integrate horizontally.

**I 2.** The domain is written

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, 2 - 2y \leq x \leq 2\},$$

because  $y \in [0, 1]$  is now the *outer* variable of integration (lying between two constants), and where  $2 - 2y \leq x \leq 2$  for the *inner* variable of integration  $x$  for any fixed  $y$ .

The double integral becomes here

$$(3) \int_B xy \, dS = \int_0^1 \left\{ \int_{2-2y}^2 xy \, dx \right\} dy = \int_0^1 \left\{ \int_{2-2y}^2 x \, dx \right\} dy.$$

Let us first calculate the inner integral,

$$\begin{aligned} \int_{2(1-y)}^2 x \, dx &= \left[ \frac{1}{2}x^2 \right]_{2(1-y)}^2 = \frac{1}{2} \cdot 2^2 \{1^2 - (1 - y)^2\} = 2(2y - y^2) \\ &= 4y - 2y^2. \end{aligned}$$

When this result is put into (3), we get by another calculation

$$\begin{aligned} \int_B xy \, dS &= \int_0^1 y(4y - 2y^2) \, dy = \int_0^1 (4y^2 - 2y^3) \, dy \\ &= \left[ \frac{4}{3}y^3 - \frac{1}{2}y^4 \right]_0^1 = \frac{5}{6}. \quad \diamond \end{aligned}$$

**Example 3.2**

A. Calculate  $\int_B x \exp(y^3) dS$ , where  $B$  is given on the figure for  $a = 1$ .

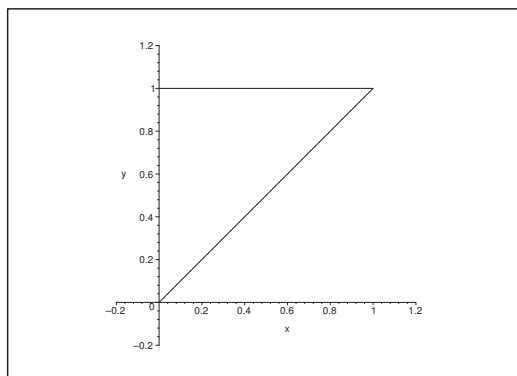


Figure 22: The domain  $B$  for  $a = 1$ .

D. We shall again examine the two possibilities of the order of integrations.

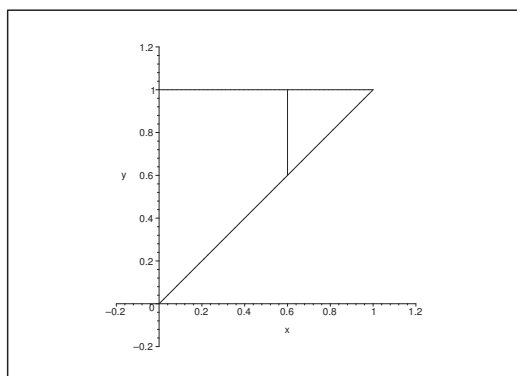


Figure 23: The domain  $B$  for  $a = 1$  with a vertical line of integration from  $y = x$  to  $y = 1$ .

**D 1.** Let us first try to integrate vertically for fixed  $x$ , just to see what happens.

**I 1.** Here the domain is written (note the order of  $x$  and  $y$ ):

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a, x \leq y \leq a\}.$$

Then we can write down the double integral:

$$\int_B x \exp(y^3) dS = \int_0^a \left\{ \int_x^a x \exp(y^3) dy \right\} dx = \int_0^a x \left\{ \int_x^a \exp(y^3) dy \right\} dx.$$

The inner integral,

$$\int_x^a \exp(y^3) dy,$$



does not just *look* impossible to calculate; it *is* impossible to calculate with our arsenal of functions! Therefore we give up this variant. Instead we examine, if we shall be more successful by interchanging the order of integration.

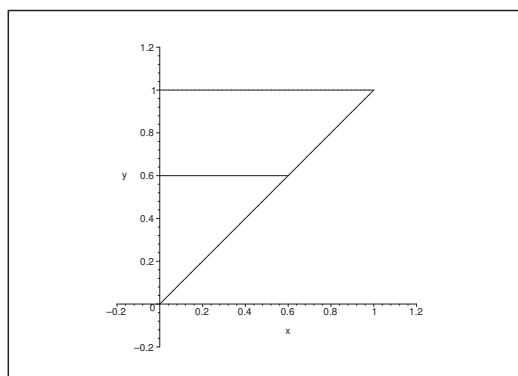


Figure 24: The domain  $B$  for  $a = 1$  with a horizontal line of integration from  $x = 0$  to  $x = y$  where  $y$  is kept fixed.

**D 2.** The domain is here written (note again the order of  $x$  and  $y$ ):

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq a, 0 \leq x \leq y\}.$$

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Then we turn to the double integral,

$$(4) \int_B x \exp(y^3) dS = \int_0^a \left\{ \int_0^y x \exp(y^3) dx \right\} dy = \int_0^a \exp(y^3) \left\{ \int_0^y x dx \right\} dy.$$

The inner integral is calculated in the following way:

$$\int_0^y x dx = \left[ \frac{1}{2} x^2 \right]_0^y = \frac{1}{2} y^2.$$

By insertion in (4) followed by the substitution  $t = y^3$  and  $dt = 3y^2 dy$ , where  $y^2 dy$  already can be found in the integrand, we get

$$\begin{aligned} \int_B x \exp(y^3) dS &= \int_0^a \exp(y^3) \cdot \frac{1}{2} y^2 dy = \int_0^{a^3} \exp(t) \cdot \frac{1}{2} \cdot \frac{1}{3} dt \\ &= \frac{1}{6} [e^t]_0^{a^3} = \frac{1}{6} \{ \exp(a^3) - 1 \}. \end{aligned}$$

**Remark 3.1** In this case one of the two variants cannot be calculated, while the second one is easy to perform.  $\diamond$

### Example 3.3

**A.** Find the value of

$$E = Lz \int_B \frac{y}{(x^2 + y^2 + z^2)^2} dS, \quad \text{where } B = [0, a] \times [0, b] \text{ and } z > 0.$$

**D.** Here we can expect a quite a few difficulties, no matter which version we are choosing. In fact, the integrand is suited for the polar coordinates, while the domain  $B$  in the  $(x, y)$ -plane is best described in rectangular coordinates. By experience, such a mixture of polar and rectangular coordinates will always give computational problems.

Then we notice that *if* we start by first integrating after  $x$ , then we shall immediately run into troubles with this first integral

$$\int \frac{dx}{(x^2 + y^2 + z^2)^2}.$$

It is possible to go through with the calculations, but they are far from elementary. On the other hand, if we first integrate with respect to  $y$ , we shall benefit from the fact that  $y dy = \frac{1}{2} d(y^2)$ , where  $y^2$  already can be found in the integrand. For that reason we choose first to integrate with respect to  $y$ .

I. The double integral is here

$$(5) \quad E = Lz \int_B \frac{y}{(x^2 + y^2 + z^2)^2} dS = Lz \int_0^a \left\{ \int_0^b \frac{y}{(x^2 + y^2 + z^2)^2} dy \right\} dx.$$

By the calculations of the inner integral we put for convenience  $c = x^2 + z^2 =$  equal a constant, and we apply the substitution  $t = y^2$  where  $dt = 2y dy$ , and where  $y dy$  already is a factor in the integrand, i.e.  $y dy = \frac{1}{2} dt$ . Thus,

$$\begin{aligned} \int_0^b \frac{y}{(x^2 + y^2 + z^2)^2} dy &= \int_0^b \frac{1}{y^2 + c)^2} y dy = \frac{1}{2} \int_0^{b^2} \frac{dt}{(t + c)^2} = \frac{1}{2} \left[ -\frac{1}{t + c} \right]_{t=0}^{b^2} \\ &= \frac{1}{2} \left( \frac{1}{c} - \frac{1}{b^2 + c} \right) = \frac{1}{2} \left( \frac{1}{x^2 + z^2} - \frac{1}{x^2 + b^2 + z^2} \right). \end{aligned}$$

The result of this calculation is then inserted into (5), by which

$$(6) \quad E = \frac{Lz}{2} \int_0^a \left( \frac{1}{x^2 + z^2} - \frac{1}{x^2 + b^2 + z^2} \right) dx.$$

Notice here that it is *not* a good idea to put everything in the same fraction with a common denominator, so we keep the form above.

A new calculation shows that if  $k^2 > 0$ , then we get by the change of variable  $t = \frac{x}{k}$  that

$$\begin{aligned} \int \frac{dx}{x^2 + k^2} &= \frac{1}{k} \int \frac{1}{1 + \left(\frac{x}{k}\right)^2} \frac{1}{k} dx = \frac{1}{k} \int \frac{dt}{1 + t^2} \\ &= \frac{1}{k} \operatorname{Arctan} t = \frac{1}{k} \operatorname{Arctan} \left( \frac{x}{k} \right). \end{aligned}$$

The trick in this type of calculation is by division to get the constant 1 in the denominator plus some square.

When this calculation is applied with

$$k_1 = z \quad \text{and} \quad k_2 = \sqrt{b^2 + z^2},$$

we get by insertion into (6) that

$$\begin{aligned} E &= \frac{Lz}{2} \left[ \frac{1}{k_1} \operatorname{Arctan} \left( \frac{x}{k_1} \right) - \frac{1}{k_2} \operatorname{Arctan} \left( \frac{x}{k_2} \right) \right]_0^a \\ &= \frac{Lz}{2} \left\{ \frac{1}{z} \operatorname{Arctan} \left( \frac{a}{z} \right) - \frac{1}{\sqrt{b^2 + z^2}} \operatorname{Arctan} \left( \frac{a}{\sqrt{b^2 + z^2}} \right) \right\} \\ &= \frac{L}{2} \left\{ \operatorname{Arctan} \left( \frac{a}{z} \right) - \frac{z}{\sqrt{b^2 + z^2}} \operatorname{Arctan} \left( \frac{a}{\sqrt{b^2 + z^2}} \right) \right\}. \quad \diamond \end{aligned}$$

### 3.2 Polar coordinates

#### Example 3.4

A. Calculate

$$I = \int_b (x^2 + y^2) dS,$$

where  $B$  is described in polar coordinates by

$$A = \left\{ (\varrho, \varphi) \mid a \leq \varrho \leq 2a, \frac{\varrho}{2a} \leq \varphi \leq \frac{\varphi}{a} \right\}.$$

Note that  $B$  has a “weird” form in  $(x, y)$ -plane, while the parameter domain  $A$  in the  $(\varrho, \varphi)$ -plane is “straightened out”, so one can apply the *rectangular* version in the  $(\varrho, \varphi)$ -plane. The price for this is that one must add the weight function  $\varrho$  to the integrand.

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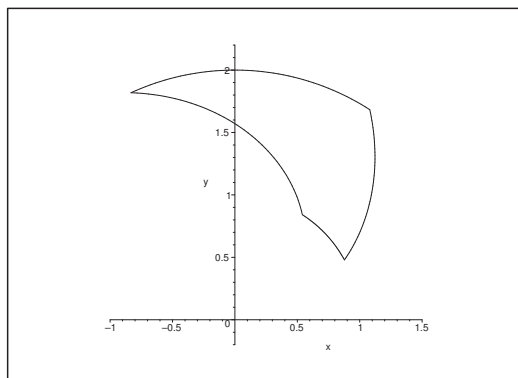


Figure 25: The domain  $B$  in the  $(x, y)$ -plane.

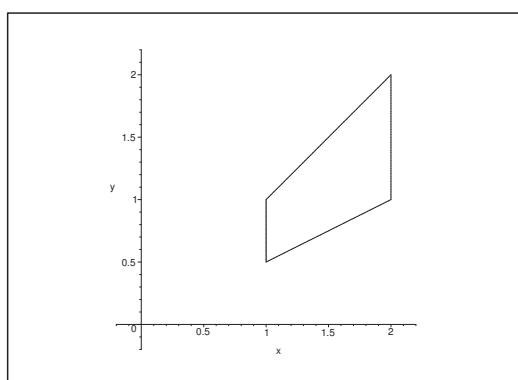


Figure 26: The parameter domain  $A$  in the  $(\varrho, \varphi)$ -plane.

**D.** Apply the reduction formula in the second version, i.e. where the  $\varphi$ -integral is the inner integral. This means that we first integrate vertically in the parameter domain.

**I.** By the reduction formula in its second version we get with the weight function  $\varrho$

$$\int_B (x^2 + y^2) dS = \int_a^{2a} \left\{ \int_{\frac{\varrho}{2a}}^{\frac{\varrho}{a}} \varrho^2 d\varphi \right\} \varrho d\varrho = \int_a^{2a} \varrho^3 \left\{ \int_{\frac{\varrho}{2a}}^{\frac{\varrho}{a}} d\varphi \right\} d\varrho.$$

First calculate the inner integral,

$$\int_{\frac{\varrho}{2a}}^{\frac{\varrho}{a}} d\varphi = \frac{\varrho}{a} - \frac{\varrho}{2a} = \frac{\varrho}{2a},$$

which is seen to be the length of the  $\varphi$ -interval.

Then by insertion,

$$\begin{aligned} \int_B (x^2 + y^2) dS &= \int_a^{2a} \varrho^3 \cdot \frac{\varrho}{2a} d\varrho = \frac{1}{2a} \int_a^{2a} \varrho^4 d\varrho = \frac{1}{2a} \left[ \frac{1}{5} \varrho^5 \right]_1^{2a} \\ &= \frac{1}{10a} \{(2a)^5 - a^5\} = \frac{1}{10a} \{32a^5 - a^5\} = \frac{31}{10} a^4. \quad \diamond \end{aligned}$$

**Example 3.5****A.** Calculate

$$I = \int_B (x + y) dS,$$

where  $B$  is described in polar coordinate (for  $a > 0$ ) by

$$A = \left\{ (\varrho, \varphi) \mid -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{4}, 0 \leq \varrho \leq a \right\},$$

i.e.  $A$  is a rectangle in the  $(\varrho, \varphi)$ -plane.

**D.** Here we can apply both reduction formulæ, so we give two solutions.**D 1.** Apply the first reduction formula; do not forget the weight function  $\varrho$ .**I 1.** From  $x = \varrho \cos \varphi$  and  $y = \varrho \sin \varphi$ , we get in the first version, where we start by integrating horizontally after  $\varrho$ , that

$$I = \int_B (x + y) dS = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \left\{ \int_0^a (\varrho \cos \varphi + \varrho \sin \varphi) \varrho d\varrho \right\} d\varphi.$$

Then calculate the inner integral,

$$\int_0^a (\varrho \cos \varphi + \varrho \sin \varphi) \varrho d\varrho = (\cos \varphi + \sin \varphi) \int_0^a \varrho^2 d\varrho = \frac{a^3}{3} (\cos \varphi + \sin \varphi).$$

By insertion of this result we finally get

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{a^3}{3} (\cos \varphi + \sin \varphi) d\varphi = \frac{a^3}{3} [\sin \varphi - \cos \varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{a^3}{3}.$$

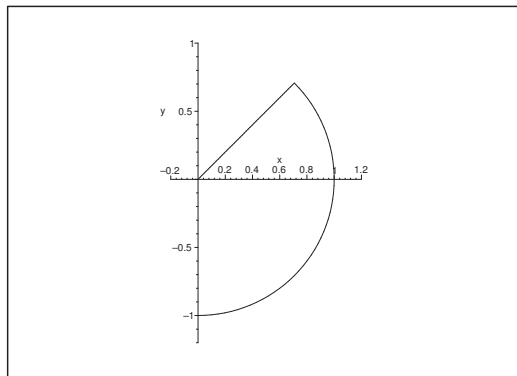


Figure 27: The domain  $B$  for  $a = 1$  in the  $(x, y)$ -plane.

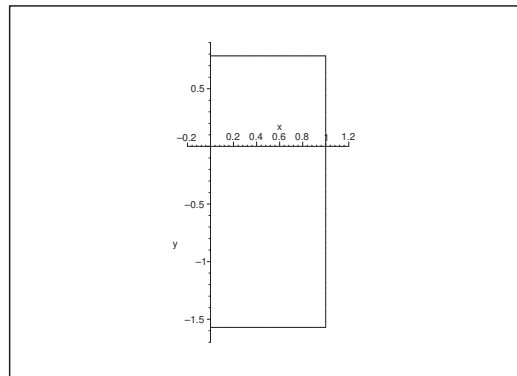


Figure 28: The parametric domain  $A$  for  $a = 1$  in the  $(\varrho, \varphi)$ -plane.


**D 2.** Apply the second reduction formula. Again, do not forget the weight function  $\varrho$ .

**I 2.** In the second version we just interchange the order of integration. Since the bounds are constants, and the variables can be separated in the integrand, we can split the integral into a product of two integrals. Then

$$\begin{aligned}
 I &= \int_0^a \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} (\varrho \cos \varphi + \varrho \sin \varphi) d\varphi \right\} \varrho d\varrho \\
 &= \int_0^a \varrho^2 d\varrho \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} (\cos \varphi + \sin \varphi) d\varphi = \frac{a^3}{3} \cdot [\sin \varphi - \cos \varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{a^3}{3}. \quad \diamond
 \end{aligned}$$


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**Example 3.6**

A. Calculate  $I = \int_B x \, dS$ , where  $B = K\left(\left(\frac{a}{2}, 0\right); \frac{a}{2}\right)$ ,  $a > 0$ .

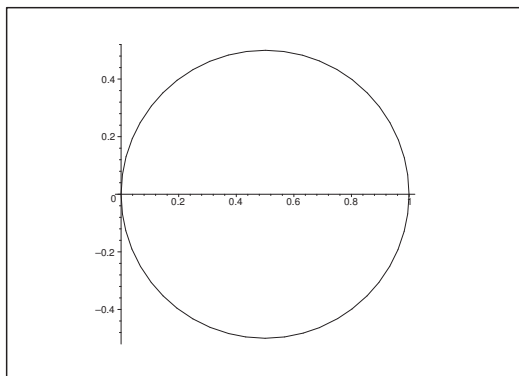


Figure 29: The domain  $B$  for  $a = 1$ , i.e.  $-\sqrt{x - x^2} \leq y \leq \sqrt{x - x^2}$  for  $0 \leq x \leq 1$ .

D. In this case it is possible to calculate the integral by using either rectangular or polar coordinates.

D 1. In rectangular coordinates the domain  $B$  is described by

$$B = \{(x, y) \mid 0 \leq x \leq a, -\sqrt{ax - x^2} \leq y \leq \sqrt{ax - x^2}\}.$$

I 1. The *rectangular* double integral is given by

$$I = \int_B x \, dS = \int_0^a x \left\{ \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy \right\} dx = \int_0^a 2x \sqrt{ax - x^2} \, dx.$$

The trick in problems of this type is to call the “ugly” part something different. We put

$$t = ax - x^2, \quad dt = (a - 2x) \, dx.$$

Then by adding the right term and subtract it again we get

$$\begin{aligned} I &= \int_0^a 2x \sqrt{ax - x^2} \, dx = - \int_0^a (a - 2x - a) \sqrt{ax - x^2} \, dx \\ &= - \int_0^a \sqrt{ax - x^2} \cdot (a - 2x) \, dx + a \int_0^a \sqrt{ax - x^2} \, dx \\ &= - \int_{x=0}^a \sqrt{t} \, dt + a \int_0^a \sqrt{ax - x^2} \, dx \\ &= - \left[ \frac{2}{3} (ax - x^2)^{\frac{3}{2}} \right]_0^a + a \int_0^a \sqrt{ax - x^2} \, dx = 0 + a \int_0^a \sqrt{ax - x^2} \, dx. \end{aligned}$$

The integral  $\int_0^a \sqrt{ax - x^2} \, dx$  does not look nice; but the geometrical interpretation helps a lot: The integral is the area of the domain between the  $x$ -axis and the curve

$$y = +\sqrt{ax - x^2},$$



i.e. (cf. the figure) the area of a half-disc of radius  $\frac{a}{2}$ . Therefore,

$$I = a \cdot \left\{ \frac{1}{2} \cdot \pi \left( \frac{a}{2} \right)^2 \right\} = \frac{\pi a^3}{8}.$$

**D 2.** The polar version; do not forget the weight function  $\varrho$ .

**I 2.** When we put  $x = \varrho \cos \varphi$  and  $y = \varrho \sin \varphi$ , the equation of the boundary curve becomes

$$0 = x^2 + y^2 - ax = \varrho^2 - a \varrho \cos \varphi = \varrho(\varrho - a \cos \varphi).$$

Since  $\varrho = 0$  corresponds to the point  $(0, 0)$ , it follows that the boundary curve is described by

$$\varrho = a \cos \varphi \quad \text{med} \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}.$$

The parametric domain  $A$  corresponding to  $B$  is therefore

$$A = \left\{ (\varrho, \varphi) \mid -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}, 0 \leq \varrho \leq a \cos \varphi \right\}.$$

When we use the first version of the reduction formula we get

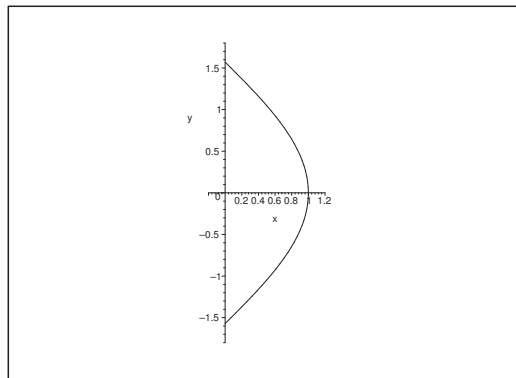


Figure 30: The parametric domain  $A$  in the  $(\varrho, \varphi)$ -plane.

$$\int_B x \, dS = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{a \cos \varphi} \varrho \cos \varphi \cdot \varrho \, d\varrho \right\} d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi \left\{ \int_0^{a \cos \varphi} \varrho^2 \, d\varrho \right\} d\varphi.$$

When we calculate the inner integral we get

$$\int_0^{a \cos \varphi} \varrho^2 \, d\varrho = \left[ \frac{1}{3} \varrho^3 \right]_0^{a \cos \varphi} = \frac{a^3}{3} \cos^3 \varphi.$$

Then by insertion

$$\int_B x \, dS = \frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \varphi \, d\varphi = 2 \cdot \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \cos^4 \varphi \, d\varphi,$$

where we use that the even function  $\cos^4 \varphi$  is integrated over a symmetric interval.

When we shall calculate a trigonometric integral, where the integrand is of *even* order, we change variables to the double angle:

$$\begin{aligned} \cos^4 x &= \{\cos^2 x\}^2 = \left\{ \frac{1}{2} (1 + \cos 2x) \right\}^2 = \frac{1}{4} \{1 + 2 \cos 2x + \cos^2 2x\} \\ &= \frac{1}{4} \left\{ 1 + 2 \cos 2x + \frac{1}{2} (1 + \cos 4x) \right\} \\ &= \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x. \end{aligned}$$

Finally, by insertion,

$$\int_B x dS = \frac{2}{3} a^3 \int_0^{\frac{\pi}{2}} \left\{ \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \right\} dx = \frac{2}{3} a^3 \cdot \frac{3}{8} \frac{\pi}{2} + 0 + 0 = \frac{\pi a^3}{8}. \quad \diamond$$


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## 4 The space integral

### 4.1 Rectangular coordinates

#### Example 4.1

A. Calculate the space integral

$$I = \int_A (3 + y - z) x \, d\Omega,$$

where

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in B, 0 \leq z \leq 2y\},$$

and  $B$  is the upper triangle shown on the figure.

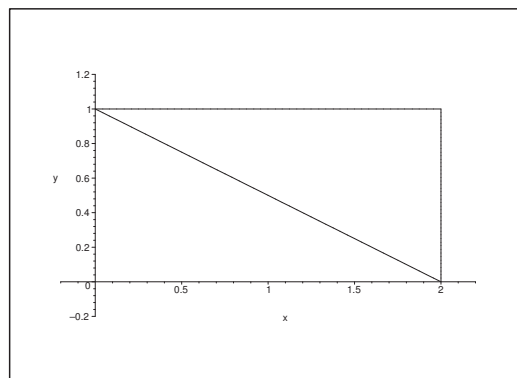


Figure 31: The domain  $B$ , i.e. the projection of the domain  $A$  onto the  $(x, y)$ -plane.

D Apply the first rectangular reduction theorem in 3 dimensions.

We have according to the first rectangular reduction theorem that

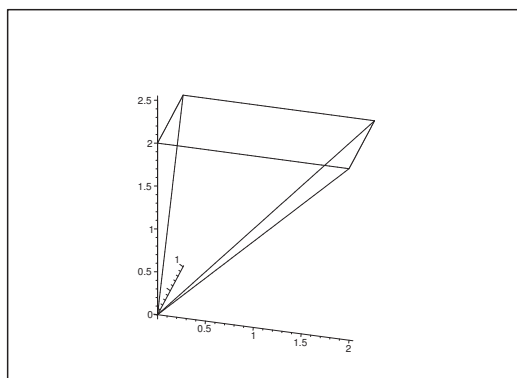
$$(7) \quad I = \int_A (3 + y - z) x \, d\Omega = \int_B \left\{ \int_0^{2y} (3 + y - z) \, dz \right\} dS.$$

Considering  $x$  and  $y$  as constants, we calculate the inner and concrete integral,

$$\begin{aligned} \int_0^{2y} (3 + y - z) x \, dz &= x \int_0^{2y} (3 + y - z) \, dz = x \left[ (3 + y)z - \frac{1}{2} z^2 \right]_{z=0}^{2y} \\ &= x \{ (3 + y) \cdot 2y - 2y^2 \} = x \cdot 2y \{ 3 + y - y \} = 6xy. \end{aligned}$$

When this result is inserted into (7), it is reduced to an abstract plane integral over  $B$ , i.e. of a lower dimension,

$$I = \int_B 6xy \, dS = 6 \int_B xy \, dS.$$

Figure 32: The domain  $A$ .

We have already calculated the abstract plane integral in Example 3.1,

$$\int_B xy \, dS = \frac{5}{6},$$

hence

$$I = \int_A (3 + y - z)x \, d\Omega = 6 \int_B xy \, dS = 6 \cdot \frac{5}{6} = 5. \quad \diamond$$

#### Example 4.2

**A.** Calculate the space integral

$$\int_A (x + 2y + z) \exp(z^4) \, d\Omega,$$

where

$$A = \{(x, y, z) \mid z \in [2, 0], (x, y) \in B(z)\}$$

with a cut at the height  $z$ ,

$$B(z) = [0, z] \times \left[0, \frac{z}{2}\right], \quad z \in ]0, 2].$$

**D.** Apply the second reduction theorem in 3 dimensions.

**I.** When we insert into the second reduction theorem, we get

$$\begin{aligned} (8) \, I &= \int_A (x + 2y + z) \exp(z^4) \, d\Omega \\ &= \int_0^2 \exp(z^4) \left\{ \int_{B(z)} (x + 2y + z) \, dS \right\} dz. \end{aligned}$$

For every fixed  $z$  we reduce the inner *abstract* plane integral,

$$\begin{aligned} \int_{B(z)} (x + 2y + z) dS &= \int_{B(z)} x dS + \int_{B(z)} 2y dy + z \int_{B(z)} dS \\ &= \int_0^z x dx \cdot \int_0^{\frac{z}{2}} dy + \int_0^z dx \cdot \int_0^{\frac{z}{2}} 2y dy + z \cdot \text{areal } B(z) \\ &= \frac{z^2}{2} \cdot \frac{z}{2} + z \cdot \frac{z^2}{4} + z \cdot \left\{ z \cdot \frac{1}{2} z \right\} = z^3. \end{aligned}$$

By insertion of this result into (8), using the substitution

$$t = z^4, \quad dt = 4z^3 dz, \quad \text{dvs.} \quad z^3 dz = \frac{1}{4} dt,$$

we finally get

$$I = \int_0^2 \exp(z^4) \cdot z^3 dz = \int_0^{2^4} e^t \cdot \frac{1}{4} dt = \frac{1}{4} (e^{16} - 1). \quad \diamond$$

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### 4.2 Semi-polar coordinates

#### Example 4.3

A. Calculate the space integral

$$I = \int_A x^2 y z \, d\Omega,$$

where

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq a^2, y \geq 0, 0 \leq z \leq h\}.$$

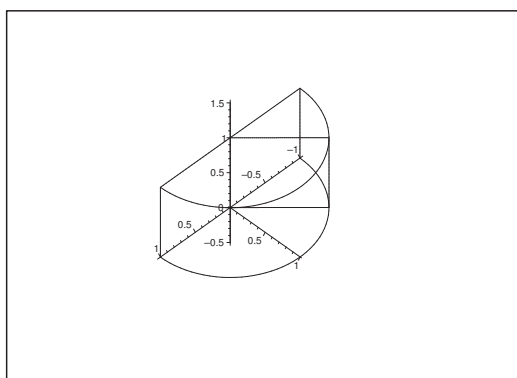


Figure 33: The domain  $A$  for  $a = h = 1$  with a cut  $B(\varphi)$ .

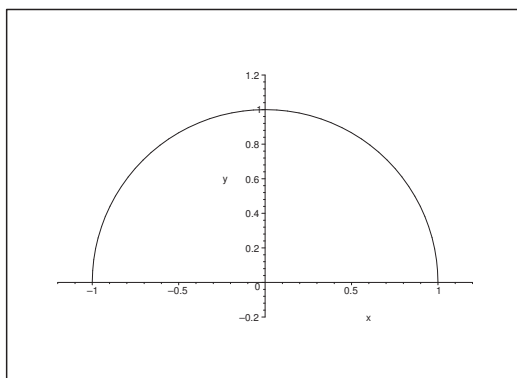


Figure 34: The projection of  $A$  onto the  $(x, y)$ -plane ( $a = 1$ ).

D. It is here possible to go through with the rectangular calculations, but we end up with the same problems as in Example 3.6. We therefore here choose the *semi-polar* representation, where we *must not forget* to add the weight function  $\rho$  as a factor.

When we use semi-polar coordinates, the domain  $A$  is represented by the parametric domain

$$\tilde{A} = \{(\varrho, \varphi, z) \mid 0 \leq \varrho \leq a, 0 \leq \varphi \leq \pi, 0 \leq z \leq h\}.$$

Then we get at least two possibilities for the reduction.

**I 1.** For fixed  $\varphi$  the domain  $A$  is cut into  $B(\varphi) = [0, a] \times [0, h]$ . In this case we get the following reduction where  $\varphi$  is kept in the outer integral,

$$\begin{aligned} I &= \int_0^\pi \left\{ \int_{B(\varphi)} z \cdot \varrho^2 \cos^2 \varphi \cdot \varrho \sin \varphi \cdot \varrho d\varrho \right\} d\varphi \\ &= \int_0^\pi \cos^2 \varphi \sin \varphi d\varphi \cdot \int_0^h z dz \cdot \int_0^a \varrho^4 d\varrho \\ &= \left[ -\frac{1}{3} \cos^3 \varphi \right]_0^\pi \cdot \left[ \frac{1}{2} z^2 \right]_0^h \cdot \left[ \frac{1}{5} \varrho^5 \right]_0^a = \frac{2}{3} \cdot \frac{1}{2} h^2 \cdot \frac{1}{5} a^5 = \frac{1}{15} h^2 a^5. \end{aligned}$$

**I 2.** If we instead integrate first after  $z$ , then we get where we use that  $B$  is a half disc,

$$\begin{aligned} I &= \int_B x^2 y \left\{ \int_0^h z dz \right\} dS \\ &= \left[ \frac{1}{2} z^2 \right]_0^h \cdot \int_0^\pi \left\{ \int_0^a \varrho^2 \cos^2 \varphi \cdot \varrho \sin \varphi \cdot \varrho d\varrho \right\} d\varphi \\ &= \frac{1}{2} h^2 \int_0^\pi \cos^2 \varphi \sin \varphi d\varphi \cdot \int_0^a \varrho^4 d\varrho \\ &= \frac{1}{2} h^2 \left[ -\frac{1}{3} \cos^3 \varphi \right]_0^\pi \cdot \frac{a^5}{5} = \frac{1}{15} h^2 a^5. \end{aligned}$$

**C.** *Weak control* (considerations of the dimension). Since

$$x \sim a, \quad y \sim a, \quad z \sim h, \quad \int \dots d\Omega = \int \dots dx dy dz \sim a \cdot a \cdot h = a^2 h$$

we get

$$I = \int_A x^2 y^2 z d\Omega \sim a^2 \cdot a^2 \cdot h \cdot (a^2 h) = h^2 a^5,$$

hence the result *must* be of the form  $\text{constant} \cdot h^2 a^5$ . If this is not the case, we have made an error. On the other hand, even if we get a result like  $c \cdot h^2 a^5$ , the constant  $c$  may still be calculated wrongly, explaining why the method is only giving a weak control.  $\diamond$

**Example 4.4**

A. Calculate the space integral

$$I = \int_A xy^2z \, d\Omega,$$

where

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq a^2, x \geq 0, \sqrt{x^2 + y^2} \leq z \leq a\}.$$

By considering the dimensions we get  $x, y, z \sim a$  and  $\int \dots d\Omega \sim a^3$ , so

$$I = \int_A xy^2z \, d\Omega \sim \cdot a^2 \cdot a \cdot a^3 = a^7.$$

Therefore, the result *must* be of the form constant  $\cdot a^7$ .

D. The shape of  $A$  (as a part of a body of revolution) is an invitation to use *semi-polar* coordinates (it cannot be said too often: *Do not forget the weight function  $\rho!$* ), where  $A$  is represented by

$$\tilde{A} = \left\{ (\varrho, \varphi, z) \mid 0 \leq \varrho \leq a, -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}, \varrho \leq z \leq a \right\}.$$

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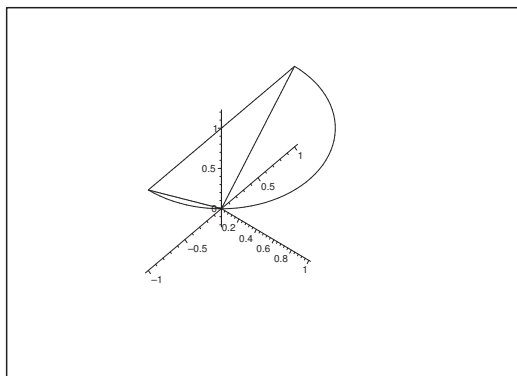


Figure 35: The domain  $A$  for  $a = 1$ .

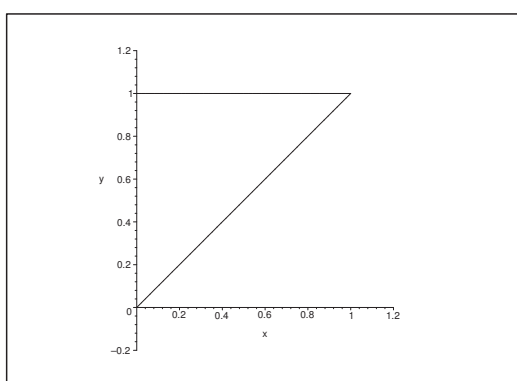


Figure 36: The cut in the meridian half-plane for  $a = 1$ .

**I.** The cut  $B(\varphi)$ , which is revolved around the  $z$ -axis, must be independent of  $\varphi$ , so have in the meridian half plane

$$B(\varphi) = \{(\varrho, z) \mid 0 \leq z \leq a, 0 \leq \varrho \leq z\}.$$

Then we get by the reduction theorem that the  $\varphi$ -integral can be factored out,

$$\begin{aligned} I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_{B(\varphi)} \varrho \cos \varphi \cdot \varrho^2 \sin^2 \varphi \cdot z \cdot \varrho d\varrho dz \right\} d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \varphi \cdot \cos \varphi d\varphi \cdot \int_{B(\varphi)} \varrho^4 z d\varrho dz \\ &= \left[ \frac{1}{3} \sin^3 \varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \int_0^a z \left\{ \int_0^z \varrho^4 d\varrho \right\} dz = \frac{2}{3} \cdot \int_0^a z \cdot \frac{1}{5} z^5 dz \\ &= \frac{2}{15} \int_0^a z^6 dz = \frac{2}{15} \cdot \frac{1}{7} a^7 = \frac{2}{105} a^7. \end{aligned}$$

**C.** It is seen as a weak control that the result is of the form constant  $\cdot a^7$  as mentioned in **A**.  $\diamond$

### 4.3 Spherical coordinates

#### Example 4.5

A. Let  $A$  be an upper half sphere of radius  $2a$ , from which we have removed a cylinder of radius  $a$  and then halved the resulting domain by the plane  $x + y = 0$ . We shall only consider that part for which  $x + y \geq 0$ . Calculate the space integral

$$\int_A xz \, d\Omega.$$

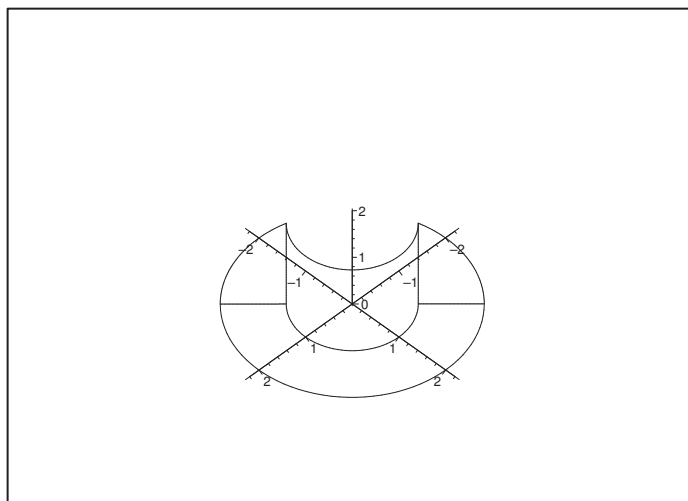


Figure 37: The domain  $A$  for  $a = 1$  in the  $(x, y, z)$ -space.

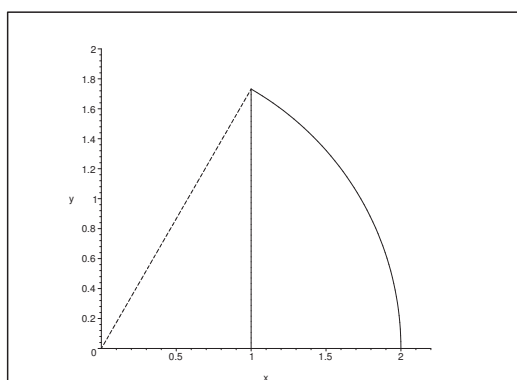


Figure 38: The cut in the meridian half-plane for  $a = 1$ , i.e. in the  $(\varrho, z)$ -half-plane.

When we consider the dimensions (i.e. a rough overview) we get

$$x \sim a, \quad y \sim a, \quad z \sim a, \quad \int_A \dots d\Omega \sim a^3,$$

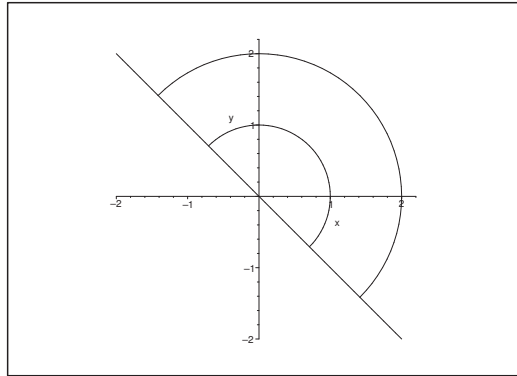


Figure 39: The projection of  $A$  onto the  $(x, y)$ -plane for  $a = 1$ .

from which  $\int_A xz \, d\Omega \sim a^5$ , and thus

$$\int_A xz \, d\Omega = \text{constant} \cdot a^5.$$

**D.** The geometrical structure of revolution and the sphere indicate that one either should apply **I 1.** *semi-polar coordinates* or **I 2.** *spherical coordinates*. We shall in the following go through both possibilities for comparison.

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**I 1.** In *semi-polar* coordinates the domain  $A$  is represented by

$$\tilde{A} = \left\{ (\varrho, \varphi, z) \mid a \leq \varrho \leq 2a, -\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, 0 \leq z \leq \sqrt{4a^2 - \varrho^2} \right\}.$$

Hence by the reduction theorem (where the weight function is  $\varrho$ ),

$$\begin{aligned} I &= \int_A xz \, d\Omega = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left\{ \int_a^{2a} \left\{ \int_0^{\sqrt{4a^2 - \varrho^2}} \varrho \cos \varphi \cdot z \, dz \right\} \varrho \, d\varrho \right\} d\varphi \\ &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos \varphi \, d\varphi \cdot \int_a^{2a} \varrho^2 \left\{ \int_0^{\sqrt{4a^2 - \varrho^2}} z \, dz \right\} d\varrho \\ &= [\sin \varphi]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \int_a^{2a} \varrho^2 \left[ \frac{1}{2} z^2 \right]_0^{\sqrt{4a^2 - \varrho^2}} d\varrho = \sqrt{2} \cdot \frac{1}{2} \int_a^{2a} \varrho^2 (4a^2 - \varrho^2) \, d\varrho \\ &= \frac{\sqrt{2}}{2} \int_a^{2a} (4a^2 \varrho^2 - \varrho^4) \, d\varrho = \frac{\sqrt{2}}{2} \left[ \frac{4}{3} a^2 \varrho^3 - \frac{1}{5} \varrho^5 \right]_a^{2a} \\ &= \frac{\sqrt{2}}{2} \left\{ \left( \frac{4}{3} 3a^2 \cdot 8a^3 - \frac{32}{5} a^5 \right) - \left( \frac{4}{3} a^2 \cdot a^3 - \frac{1}{5} a^5 \right) \right\} \\ &= \frac{\sqrt{2}}{2} a^5 \left\{ \frac{32}{3} - \frac{32}{5} - \frac{4}{3} + \frac{1}{5} \right\} \\ &= \frac{\sqrt{2}}{2} a^5 \cdot \left\{ \frac{28}{3} - \frac{31}{5} \right\} = \frac{47\sqrt{2} a^5}{30}. \end{aligned}$$

**I 2.** If we instead choose *spherical* coordinates then

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta,$$

where  $\theta$  is *measured from the  $z$ -axis* (and *not* from the  $(x, y)$ -plane, which one might expect), and the weight function is  $r^2 \sin \theta$ , and the domain  $A$  is represented by the parametric space

$$\hat{A} = \left\{ (r, \varphi, \theta) \mid -\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}, \frac{a}{\sin \theta} \leq r \leq 2a \right\},$$

where the vertical bounding line for  $B_0$  is described by  $r \sin \theta = a$ , so the lower bound for  $r$  is  $\frac{a}{\sin \theta} \leq r$ .

Then we get by the reduction theorem

$$\begin{aligned}
 \int_A xz \, d\Omega &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left\{ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left\{ \int_{\frac{a}{\sin \theta}}^{2a} r \sin \theta \cos \varphi \cdot r \cos \theta r^2 \sin \theta \, dr \right\} d\theta \right\} d\varphi \\
 &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos \varphi \, d\varphi \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 \theta \cos \theta \left\{ \int_{\frac{a}{\sin \theta}}^{2a} r^4 \, dr \right\} d\theta \\
 &= [\sin \varphi]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 \theta \cdot \cos \theta \left[ \frac{1}{5} r^5 \right]_{\frac{a}{\sin \theta}}^{2a} d\theta \\
 &= \sqrt{2} \cdot 5 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 \theta \cos \theta \cdot \left\{ 32 a^5 - \frac{a^5}{\sin^5 \theta} \right\} d\theta \\
 &= \frac{\sqrt{2}}{5} a^5 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left\{ 32 \sin^2 \theta - \frac{1}{\sin^3 \theta} \right\} \cos \theta \, d\theta \\
 &= \frac{\sqrt{2}}{5} a^5 \left[ \frac{32}{3} \sin^3 \theta + \frac{1}{2} \frac{1}{\sin^2 \theta} \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
 &= \frac{\sqrt{2}}{5} a^5 \left\{ \left( \frac{32}{3} + \frac{1}{2} \right) - \left( \frac{32}{3} \cdot \frac{1}{8} + \frac{1}{2} \cdot 4 \right) \right\} \\
 &= \frac{\sqrt{2}}{5} a^5 \left\{ \frac{32}{3} + \frac{1}{2} - \frac{4}{3} - 2 \right\} = \frac{47\sqrt{2}}{60} a^5.
 \end{aligned}$$

C. We see in both variants that the result is  $\sim a^5$ , so we get a weak control, cf. the examination of the dimensions in A.  $\diamond$

**Example 4.6**

A. Let  $A$  be the spindle formed domain on the figure, which is obtained by revolving the figure in the meridian half-plane around the  $z$ -axis. Calculate the space integral  $I = \int_A z \, d\Omega$ .

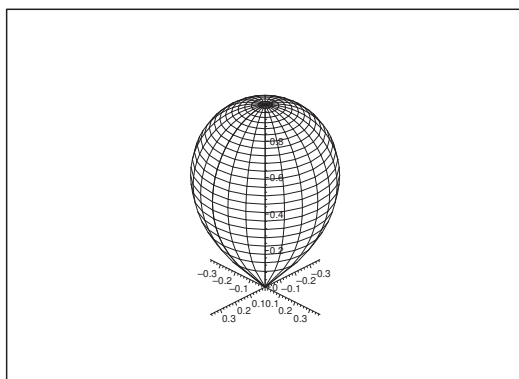


Figure 40: The domain  $A$  for  $a = 1$  in the  $(x, y, z)$ -space.

An examination of the dimensions gives  $z \sim a$  and  $\int_A \dots d\Omega \sim a^3$ , hence

$$I = \int_A z \, d\Omega = c \cdot a \cdot a^3 = c \cdot a^4,$$

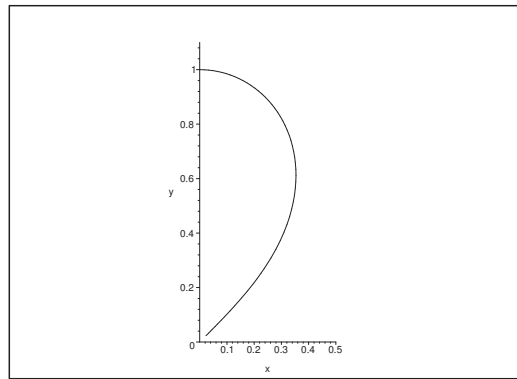


Figure 41: The meridian half-plane with the curve  $r = \sqrt{\cos 2\theta}$  for  $a = 1$  given spherically, i.e.  $r$  is the distance from  $(0, 0)$  to a point on the curve.

where the task now is to find  $c$ .

**D.** Since  $A$  is a domain of revolution it is natural either to choose *spherical* or *semi-polar* coordinates. In this particular case is the variant **I 1**. *Spherical coordinates* the easiest one to apply. We have for comparison added **I 2**, so one can see what may happen if one chooses to apply one method on all problems.

**I 1.** We have in *spherical coordinates* that

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta,$$

represent  $A$  by the parametric domain (cf. the figure over the meridian half-plane)

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$$\tilde{A} = \{(r, \varphi, \theta) \mid 0 \leq \varphi \leq 2\pi, 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq a\sqrt{\cos 2\theta}\}.$$

Since neither the bounds for  $\theta$  or  $r$ , the weight function  $r^2 \sin \theta$  or  $z = r \cos \theta$  contain the variable  $\varphi$ , we can factorize the integrations

$$\int_0^{2\pi} 1 \, d\varphi = 2\pi,$$

hence we get by the reduction theorem,

$$\begin{aligned} I &= 2\pi \int_0^{\frac{\pi}{4}} \left\{ \int_0^{a\sqrt{\cos 2\theta}} r \cos \theta \cdot r^2 \sin \theta \, dr \right\} d\theta \\ &= 2\pi \int_0^{\frac{\pi}{4}} \cos \theta \sin \theta \left\{ \int_0^{a\sqrt{\cos 2\theta}} r^3 \, dr \right\} d\theta. \end{aligned}$$

We first calculate the inner integral, where  $\theta$  is considered as a constant,

$$\int_0^{a\sqrt{\cos 2\theta}} r^3 \, dr = \left[ \frac{1}{4} r^4 \right]_0^{a\sqrt{\cos 2\theta}} = \frac{a^4}{4} \cos^2 2\theta.$$

Then by insertion,

$$\begin{aligned} I &= 2\pi \int_0^{\frac{\pi}{4}} \cos \theta \sin \theta \cdot \frac{a^4}{4} \cos^2 2\theta \, d\theta \\ &= \frac{\pi a^4}{4} \int_0^{\frac{\pi}{4}} \cos^2 2\theta \sin 2\theta \, d\theta = \frac{\pi a^4}{4} \int_{\theta=0}^{\frac{\pi}{4}} \cos^2 2\theta \left( -\frac{1}{2} \right) d \cos 2\theta \\ &= -\frac{\pi a^4}{8} \left[ \frac{1}{3} \cos^3 2\theta \right]_0^{\frac{\pi}{4}} = \frac{\pi a^4}{24} \{1 - 0\} = \frac{\pi a^4}{24}. \end{aligned}$$

**I 2.** Let us now turn to the *semi-polar* variant. The problem with this is to find  $\varrho = P(z)$  as a function of  $z$  for the boundary curve  $r = a\sqrt{\cos 2\theta}$  for the domain in the meridian half-plane.

When we express  $\varrho$  by means of  $r$  and  $\theta$  we get (cf. e.g. the meridian-half-plane)

$$\varrho = r \sin \theta \quad \text{and} \quad z = r \cos \theta. \quad \text{hence} \quad r^2 = \varrho^2 + z^2.$$

Then the task is to eliminate  $r$  and  $\theta$  from  $r = a\sqrt{\cos 2\theta}$ . By squaring this equation we get

$$r^2 = a^2 \cos 2\theta = a^2 (\cos^2 \theta - \sin^2 \theta).$$

It follows from the expressions of  $\varrho$  and  $z$  that we can get rid of  $\cos^2 \theta$  and  $\sin^2 \theta$  by multiplying by  $r^2$ , i.e.

$$(r^2)^2 = (\varrho^2 + z^2)^2 = a^2 (r^2 \cos^2 \theta - r^2 \sin^2 \theta) = a^2 z^2 - a^2 \varrho^2.$$

Since  $r \geq 0$  and  $\varrho \geq 0$ , these two operations are “equivalent”, i.e. we have not obtained some further “false solutions”.

When the equation above is rearranged we get an equation of second order in  $\varrho^2$ ,

$$(\varrho^2)^2 + (2z^2 + a^2) \varrho^2 + z^4 - a^2 z^2 = 0.$$

This is solved in high school manner, where we just put + in front of the square root, because  $\varrho^2 > 0$ ,

$$\begin{aligned} \varrho^2 &= \frac{1}{2} \left\{ -(2z^2 + a^2) + \sqrt{(2z^2 + a^2)^2 - 4z^4 + 4z^2 a^2} \right\} \\ &= \frac{1}{2} \left\{ \sqrt{4z^4 + 4z^2 a^2 + a^4 - 4z^4 + 4z^2 a^2} - (2z^2 + a^2) \right\} \\ &= \frac{1}{2} \left\{ a\sqrt{8z^2 + a^2} - (2z^2 + a^2) \right\}. \end{aligned}$$

A test shows that  $\varrho^2 = 0$  for  $z = 0$  and  $z = a$ , which is in harmony with the situation in the meridian half-plane.

Formally  $A$  is therefore in *semi-polar* coordinates represented by the parametric space

$$\hat{A} = \left\{ (\varrho, \varphi, z) \mid \leq \varphi \leq 2\pi, 0 \leq z \leq a, 0 \leq \varrho \leq \sqrt{\frac{1}{2} \left\{ a\sqrt{8z^2 + a^2} - (2z^2 + a^2) \right\}} \right\},$$

which does not look too nice. However, it is not as bad as it seems to be, because the integrand only depends on  $z$ . By using the “*slice method*”, we get by a reduction that

$$I = \int_A z d\Omega = \int_0^a z \left\{ \int_{B(z)} dS \right\} dz = \int_0^a z \cdot \text{areal } B(z) dz,$$

where  $B(z)$  is the disc of radius

$$\varrho = P(z) = \sqrt{\frac{1}{2} \left\{ a\sqrt{8z^2 + a^2} - (2z^2 + a^2) \right\}}.$$

Then

$$\text{area } B(z) = \pi \{P(z)\}^2 = \frac{\pi}{2} \left\{ a\sqrt{8z^2 + a^2} - (2z^2 + a^2) \right\},$$



from which

$$\begin{aligned}
 I &= \int_A z \, d\Omega = \int_0^a z \cdot \text{area } B(z) \, dz \\
 &= \frac{\pi}{2} \int_0^a \{a\sqrt{8z^2 + a^2} - 2z^2 - a^2\} z \, dz \quad (\text{substitute: } t = z^2, \, dt = 2z \, dz) \\
 &= \frac{\pi}{4} \int_0^{a^2} \{a\sqrt{8t + a^2} - 2t - a^2\} \, dt \\
 &= \frac{\pi}{4} a \int_0^{a^2} \sqrt{8t + a^2} \, dt - \frac{\pi}{4} [t^2 + a^2t]_0^{a^2} \\
 &= \frac{\pi}{a} \left[ \frac{1}{8} \cdot \frac{2}{3} (8t + a^2)^{\frac{3}{2}} \right]_0^{a^2} - \frac{\pi}{2} a^4 \\
 &= \frac{\pi}{4} a \cdot \frac{1}{12} \{27a^3 - a^3\} - \frac{\pi}{2} a^4 = \frac{\pi}{24} a^4 (13 - 12) = \frac{\pi}{24} a^4.
 \end{aligned}$$

C. *Weak control.* We see in both cases that the result is of the form  $c \cdot a^4$ , as already deduced in A.

**Remark 4.1** We see that this problem could be calculated in both spherical and semi-polar coordinates, though the variant of the semi-polar coordinates was far more difficult than the spherical version. Occasionally one may find similar problems in examinations sets, where the composer of the problem thought that it was obvious to use the spherical coordinates, while the students unfortunately preferred the semi-polar coordinates instead. This has through the years caused a lot of frustration. Therefore, try also to learn the spherical method as well as the semi-polar version, and examine the role of the geometry in the choice of method.  $\diamond$

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## 5 The line integral

### Example 5.1

A. Find the curve length from  $(0, 0)$  of any finite piece  $(0 \leq \varphi \leq \alpha)$  of the Archimedes's spiral, given in polar coordinates by

$$\varrho = a\varphi, \quad 0 \leq \varphi < +\infty, \quad \text{hvor } a > 0,$$

i.e. calculate the line integral

$$\ell = \int_{\varphi=0}^{\alpha} ds.$$

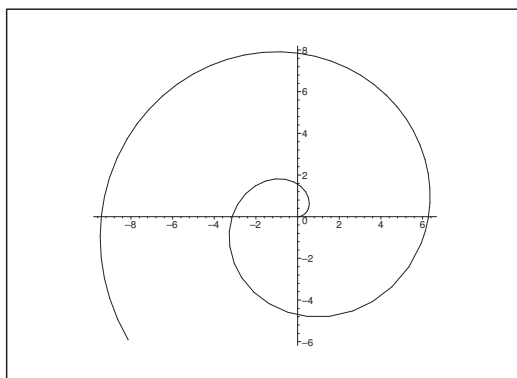


Figure 42: A piece of the Archimedes's spiral for  $a = 1$ .

D. First find the line element  $ds$  expressed by means of  $\varphi$  and  $d\varphi$ .

We shall here meet a very unpleasant integral, which we shall calculate in three different ways:

- 1) by a substitution,
- 2) by using partial integration,
- 3) by using a pocket calculator.

I. Since  $\varrho = P(\varphi) = a\varphi$ , and since we have a description of the curve in *polar* coordinates, the line element is

$$ds = \sqrt{\{P(\varphi)\}^2 + \{P'(\varphi)\}^2} d\varphi = \sqrt{(a\varphi)^2 + a^2} d\varphi = a\sqrt{1 + \varphi^2} d\varphi.$$

Then by a reduction,

$$\ell = \int_{\varphi=0}^{\alpha} ds = \int_0^{\alpha} a\sqrt{1 + \varphi^2} d\varphi = a \int_0^{\alpha} \sqrt{1 + \varphi^2} d\varphi.$$

- 1) Since  $1 + \sinh^2 t = \cosh^2 t$ , we have  $\sqrt{1 + \sinh^2 t} = +\cosh t$ , because both sides of the equation sign must be positive. Thus we can remove the square root of the integrand by using the *monotonous* substitution,

$$\varphi = \sinh t, \quad d\varphi = \cosh t dt, \quad t = \operatorname{Arsinh} \varphi = \ln \left( \varphi + \sqrt{1 + \varphi^2} \right).$$

Since  $t$  can be expressed uniquely by  $\varphi$ , the substitution must be *monotonous*.

Then

$$\begin{aligned} \ell &= a \int_0^\alpha \sqrt{1 + \varphi^2} d\varphi = a \int_{\varphi=0}^\alpha \sqrt{1 + \sinh^2 t} \cdot \cosh t dt \\ &= a \int_{\varphi=0}^\alpha \cosh^2 t dt = a \cdot \frac{1}{2} \int_{\varphi=0}^\alpha \{ \cosh 2t + 1 \} dt \\ &= \frac{a}{2} \left[ \frac{1}{2} \sinh 2t + t \right]_{\varphi=0}^\alpha = \frac{a}{2} [(t + \sinh t \cdot \cosh t)]_{\varphi=0}^\alpha \\ &= \frac{a}{2} \left[ t + \sinh t \cdot \sqrt{1 + \sinh^2 t} \right]_{\varphi=\sinh t=0}^\alpha \\ &= \frac{a}{2} \left[ \ln \left( \varphi + \sqrt{1 + \varphi^2} \right) + \varphi \cdot \sqrt{1 + \varphi^2} \right]_0^\alpha \\ &= \frac{a}{2} \left\{ \alpha \sqrt{1 + \alpha^2} + \ln \left( \alpha + \sqrt{1 + \alpha^2} \right) \right\}. \end{aligned}$$

- 2) If we instead apply *partial integration*, then

$$\begin{aligned} \ell &= a \int_0^\alpha \sqrt{1 + \varphi^2} d\varphi = a \int_0^\alpha 1 \cdot \sqrt{1 + \varphi^2} d\varphi \\ &= a \left[ \varphi \sqrt{1 + \varphi^2} \right]_0^\alpha - a \int_0^\alpha \varphi \cdot \frac{\varphi}{\sqrt{1 + \varphi^2}} d\varphi \\ &= a \left\{ \alpha \sqrt{1 + \alpha^2} - \int_0^\alpha \frac{(\varphi^2 + 1) - 1}{\sqrt{1 + \varphi^2}} d\varphi \right\} \\ &= a \left\{ \alpha \sqrt{1 + \alpha^2} - \int_0^\alpha \sqrt{1 + \varphi^2} d\varphi + \int_0^\alpha \frac{d\varphi}{\sqrt{1 + \varphi^2}} \right\} \\ &= -a \int_0^\alpha \sqrt{1 + \varphi^2} d\varphi + a \left\{ \alpha \sqrt{1 + \alpha^2} + \ln \left( \alpha + \sqrt{1 + \alpha^2} \right) \right\}. \end{aligned}$$

The first term is  $-a \int_0^\alpha \sqrt{1 + \varphi^2} d\varphi = -\ell$ , so we get by adding  $\ell$  and dividing by 2 that

$$\ell = \frac{a}{2} \left\{ \alpha \sqrt{1 + \alpha^2} + \ln \left( \alpha + \sqrt{1 + \alpha^2} \right) \right\}.$$

- 3) This is an example where a *pocket calculator* will give an equivalent, though different answer, so it is easy to see for the teacher, whether a pocket calculator has been applied or not. It is here illustrated by the use of a TI-89, where the *command* is given by

$$\mathbf{a} \star \int(\sqrt{1 + t^2}, t, 0, b),$$

because neither  $\varphi$  nor  $\alpha$  are natural. Then the answer of the pocket calculator is

$$(9) \ a \cdot \left( \frac{\ln(\sqrt{b^2 + 1} + b)}{2} + \frac{b\sqrt{b^2 + 1}}{2} \right),$$

followed by writing  $\alpha$  again instead of  $b$ .

However, if one does *not* apply a pocket calculator, but instead uses the standard methods of integration, one would *never* state the result in the form (9). The reason for this discrepancy is that the programs of the pocket calculator are created from specialists in Algebra, and they do not always speak the same mathematical language as the specialists in Calculus or Mathematical Analysis. In Calculus the priority of the terms would be  $(b = \alpha)$

$$\frac{a}{2} \left\{ \alpha \sqrt{1 + \alpha^2} + \ln \left( \alpha + \sqrt{1 + \alpha^2} \right) \right\},$$

because one would try to put as many factors as possible outside the parentheses and then order the rest of the terms, such that the simplest is also the first one. Obviously, this is not the structure of (9).

The phenomenon was discovered at an examination where pocket calculators were only allowed if one also wrote down the applied command and the type of the pocket calculator. Many students did not do it, and yet it was discovered that they had used a pocket calculator.

The morale of this story is that even if a pocket calculator may give the right result, this result does not have to be put in a practical form. It is even worse by applications of e.g. MAPLE

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where the result is sometimes given in a form using functions which are not known by students of Calculus.

Note also that pocket calculators in general do not like the operations  $|\cdot|$  and  $\sqrt{\cdot}$ , and cases where we have got two parameters. The latter is not even one of the favorites of MAPLE either, and it is in fact possible to obtain some very strange results by using MAPLE on even problems from this part of Calculus. I shall therefore warn the students: *Do not use pocket calculators and computer programs like MAPLE or Mathematica uncritically!* Since they exist, they should of course be applied, but do it with care.  $\diamond$

### Example 5.2

A. Find the value of the line integral  $I = \int_{\mathcal{K}} |y| ds$ , where  $\mathcal{K}$  is the cardioid given in polar coordinates by

$$\varrho = P(\varphi) = a(1 + \cos \varphi), \quad -\pi \leq \varphi \leq \pi.$$

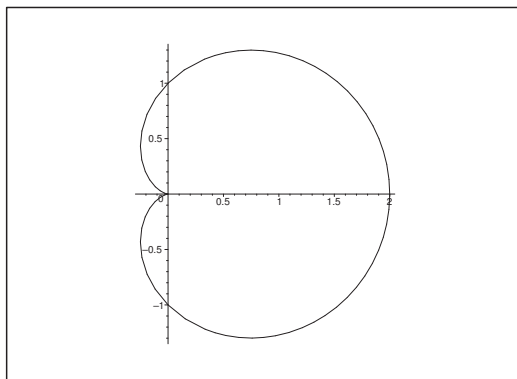


Figure 43: The cardioid for  $a = 1$ ; ( $\kappa\alpha\rho\delta\iota\alpha$  = heart).

Examination of dimensions: Since  $\varrho \sim a$ , We get  $\int_{\mathcal{K}} \cdots ds \sim a$ , and since  $y \sim a$ , The result must be of the form  $c \cdot a \cdot a = c \cdot a^2$ .

Due to the *numerical sign* in the integrand we must be very careful. In particular, a pocket calculator will be in big trouble here, if one does not give it a hand from time to time during the calculations.

D. First find the line element  $ds$ .

I. The line element is seen to be

$$\begin{aligned} ds &= \sqrt{\{P(\varphi)\}^2 + \{P'(\varphi)\}^2} d\varphi = \sqrt{\{a(1 + \cos \varphi)\}^2 + \{-a \sin \varphi\}^2} d\varphi \\ &= a\sqrt{(1 + 2 \cos \varphi + \cos^2 \varphi) + \sin^2 \varphi} d\varphi = a\sqrt{2} \cdot \sqrt{1 + \cos \varphi} d\varphi. \end{aligned}$$

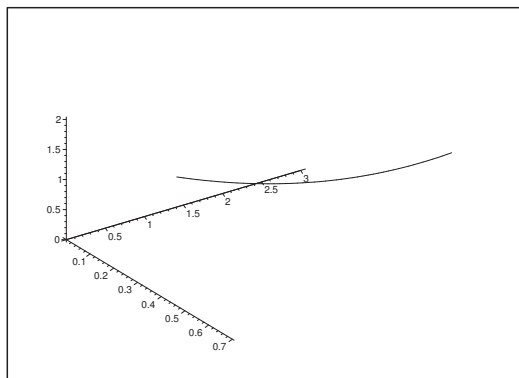


Figure 44: The space curve  $\mathbf{x} = \mathbf{r}(t)$ .

The line integral becomes

$$\begin{aligned}
 I &= \int_1^2 \frac{1}{t} \cdot \left\{ (\sqrt{2} \cdot t)^2 + 2 \left( \frac{1}{2} t^2 \right) \right\} \cdot \left( \frac{1}{t} + t \right) dt \\
 &= \int_1^2 \frac{1}{t} (2t^2 + t^2) \cdot \left( \frac{1}{t} + t \right) dt = \int_1^2 (3 + 3t^2) dt = 3 + [t^3]_1^2 = 10. \quad \diamond
 \end{aligned}$$

**Example 5.4**

**A.** Let  $a, h > 0$ . Consider the helix

$$\mathbf{r}(t) = (x, y, z) = (a \cos t, a \sin t, ht), \quad t \in \mathbb{R}.$$

This is lying on the cylinder  $x^2 + y^2 = a^2$ .

Find the natural parametric representation of the curve from  $(a, 0, 0)$ , corresponding to  $t = 0$ .

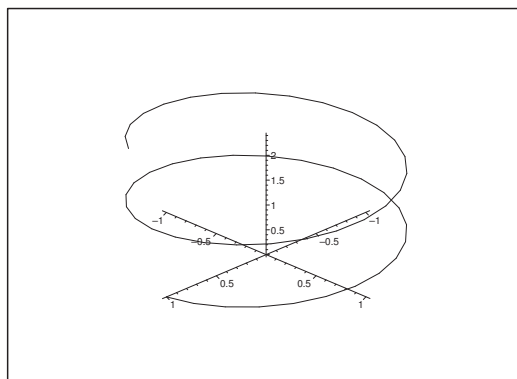


Figure 45: The helix for  $a = 1$  and  $h = \frac{1}{5}$ .

**D.** Find the arc length  $s = s(t)$  as a function of the parameter  $t$ . Solve this equation  $t = t(s)$ , and put the result into the parametric representation above.

I. Let us first find the *line element*  $ds = \|\mathbf{r}'(t)\| dt$ . Since

$$\mathbf{r}'(t) = (-a \sin t, a \cos t, h),$$

we have

$$\|\mathbf{r}'(t)\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + h^2} = \sqrt{a^2 + h^2},$$

hence the arc length is

$$s = s(t) = \int_0^t \|\mathbf{r}'(\tau)\| d\tau = \int_0^t \sqrt{a^2 + h^2} d\tau = \sqrt{a^2 + h^2} \cdot t.$$

By solving after  $t$  we get

$$t = t(s) = \frac{s}{\sqrt{a^2 + h^2}}.$$

When this is put into the parametric representation of the helix, we get

$$\begin{aligned} (x, y, z) &= (a \cos t, a \sin t, ht) \\ &= \left( a \cos \left( \frac{s}{\sqrt{a^2 + h^2}} \right), a \sin \left( \frac{s}{\sqrt{a^2 + h^2}} \right), \frac{hs}{\sqrt{a^2 + h^2}} \right), \quad s \in \mathbb{R}, \end{aligned}$$

which is the natural parametric representation of the helix.  $\diamond$

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## 6 The surface integral

### Example 6.1

A. Find the surface integral  $I = \int_{\mathcal{F}} |z| dS$ , where  $\mathcal{F}$  is given by the parametric representation

$$(x, y, z) = \mathbf{r}(u, v) = (u \sin v, u \cos v, uv) = u(\sin v, \cos v, v),$$

where  $-1 \leq u \leq 1$ ,  $0 \leq v \leq 1$ .

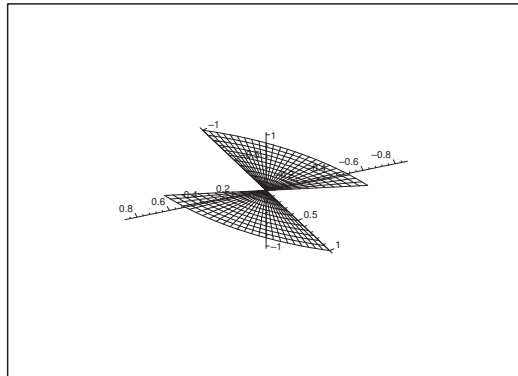


Figure 46: The surface  $\mathcal{F}$  has two components.

If we keep  $u = 1$  fixed and let  $v$  vary, then we get an arc of the helix with  $a = h = 1$ , cf. Example 5.4. When  $(0, 0, 0)$  is removed, the surface is split into its two components  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , which are symmetric with respect to the point  $(0, 0, 0)$ . The surface  $\mathcal{F}_1$  is obtained by drawing all lines from  $(0, 0, 0)$  to a point on the helix.

D. The area element is given by  $dS = \|\mathbf{N}(u, v)\| du dv$ . We first calculate the normal vector  $\mathbf{N}(u, v)$  corresponding to the given parametric representation.

I. It follows from  $\mathbf{r}(u, v) = u(\sin v, \cos v, v)$  that

$$\frac{\partial \mathbf{r}}{\partial u} = (\sin v, \cos v, v), \quad \frac{\partial \mathbf{r}}{\partial v} = u(\cos v, -\sin v, 1),$$

hence the normal vector is

$$\begin{aligned} \mathbf{N}(u, v) &= \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \sin v & \cos v & v \\ u \cos v & -u \sin v & u \end{vmatrix} \\ &= u(\cos v + v \sin v, v \cos v - \sin v, -1) \\ &= u\{(\cos v, -\sin v, -1) + v(\sin v, \cos v, 0)\}. \end{aligned}$$

Now

$$(\cos v, -\sin v, -1) \cdot (\sin v, \cos v, 0) = 0,$$



so the two vectors are perpendicular. Then we get by Pythagoras's theorem

$$\begin{aligned} \|\mathbf{N}(u, v)\|^2 &= u^2 \{ \|(\cos v, -\sin v, -1)\|^2 + v^2 \|(\sin v, \cos v, 0)\|^2 \} \\ &= u^2 \{ (\cos^2 v + \sin^2 + 1) + v^2 (\sin^2 v + \cos^2 + 0^2) \} \\ &= u^2 \{ 2 + v^2 \}. \end{aligned}$$

Notice that  $-1 \leq u \leq 1$  shows that  $u$  may be *negative*. When we take the square root we get the area element

$$dS = \|\mathbf{N}(u, v)\| du dv = |u| \sqrt{2 + v^2} du dv.$$

Putting  $D = [-1, 1] \times [0, 1]$  we get by the reduction formula

$$\begin{aligned} I &= \int_{\mathcal{F}} |z| dS = \int_D |uv| \cdot |u| \sqrt{2 + v^2} du dv \\ &= \int_{-1}^1 \left\{ \int_0^1 |u|^2 |v| \sqrt{2 + v^2} dv \right\} du = \int_{-1}^1 u^2 du \cdot \int_0^1 \sqrt{2 + v^2} \cdot v dv \\ &= \left[ \frac{1}{3} u^3 \right]_{-1}^1 \cdot \int_2^3 \sqrt{t} \cdot \frac{1}{2} dt = \frac{2}{3} \cdot \left( \frac{1}{2} \left[ \frac{2}{3} t^{\frac{3}{2}} \right]_2^3 \right) \\ &= \frac{2}{9} (3\sqrt{3} - 2\sqrt{2}). \quad \diamond \end{aligned}$$

**Example 6.2**

A. Let  $\mathcal{F}$  be the surface given by the graph representation

$$0 \leq x \leq \sqrt{3}, \quad 0 \leq y \leq \sqrt{1 + x^2}, \quad z = xy.$$

Find the surface integral  $\int_{\mathcal{F}} z dS$ .

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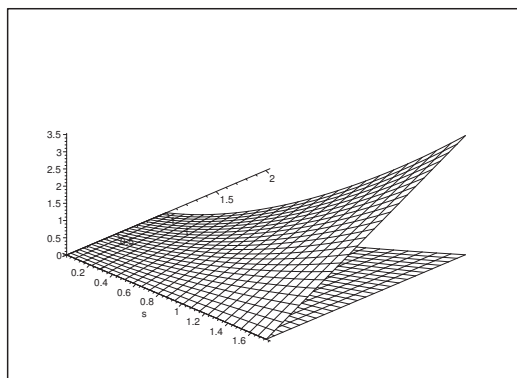
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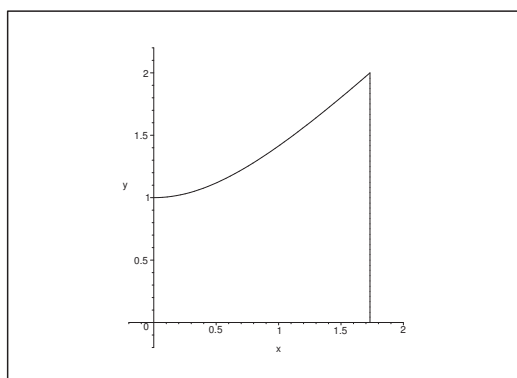
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Figure 47: The surface  $\mathcal{F}$  with its projection  $E$ .

**D.** The usual procedure is to consider  $\mathcal{F}$  as a graph of the function

$$z = f(x, y) = xy, \quad (x, y) \in E.$$

We shall not do this here, but instead *alternatively* introduce a rectangular parametric representation  $(x, y, z) = \mathbf{r}(u, v)$ . Then afterwards we shall find the weight function  $\|\mathbf{N}(u, v)\|$ .

Figure 48: The projection  $E$  of  $\mathcal{F}$  in the  $(x, y)$ -plane.

**I.** The parameters  $u$  and  $v$  are for obvious reasons not given above. They are introduced by duplicating  $(x, y)$  by the trivial formula

$$(x, y) = (u, v),$$

i.e. we choose the parametric representation

$$\mathbf{r}(u, v) = (x, y, z) = (u, v, uv), \quad 0 \leq u \leq \sqrt{3}, \quad 0 \leq v \leq \sqrt{1 + u^2},$$

so we can distinguish between  $(x, y)$  as the first two coordinates on the *surface* in the 3-dimensional space, and  $(u, v) \in E$  in the parametric domain. By experience it is always difficult to understand why we use this duplication, until one realizes that we in this way can describe two different aspects (as described above) of the same coordinates. This will be very useful in the following.

Since

$$\frac{\partial \mathbf{r}}{\partial u} = (1, 0, v) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = (0, 1, u),$$

the corresponding normal vector becomes

$$\mathbf{N}(u, v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1).$$

Hence

$$\|\mathbf{N}(u, v)\| = \sqrt{1 + u^2 + v^2}.$$

When  $dS$  denotes the area element on  $\mathcal{F}$ , and  $dS_1$  denotes the area element on  $E$ , then we have the correspondence

$$dS = \sqrt{1 + u^2 + v^2} dS_1.$$

The abstract *surface* integral over  $\mathcal{F}$  is therefore reduced to the abstract *plane* integral over  $E$  by

$$\int_{\mathcal{F}} z dS = \int_E uv \sqrt{1 + u^2 + v^2} dS.$$

Then we reduce the abstract *plane* integral over  $E$  in rectangular coordinates, where the  $v$ -integral is the inner one,

$$\int_{\mathcal{F}} z dS = \int_E uv \sqrt{1 + u^2 + v^2} dS = \int_0^{\sqrt{3}} u \left\{ \int_0^{\sqrt{1+u^2}} \sqrt{1 + u^2 + v^2} v dv \right\} du.$$

Calculate the inner integral by means of the substitution

$$t = v^2, \quad dt = 2v dv.$$

From this we get

$$\begin{aligned} \int_0^{\sqrt{1+u^2}} \sqrt{1 + u^2 + v^2} v dv &= \int_0^{1+u^2} \sqrt{1 + u^2 + t} \cdot \frac{1}{2} dt \\ &= \frac{1}{2} \left[ \frac{2}{3} (1 + u^2 + t)^{\frac{3}{2}} \right]_{t=0}^{1+u^2} = \frac{1}{3} \left\{ (2(1 + u^2))^{\frac{3}{2}} - (1 + u^2)^{\frac{3}{2}} \right\} \\ &= \frac{1}{3} (2\sqrt{2} - 1) \cdot (1 + u^2)^{\frac{3}{2}}. \end{aligned}$$

By insertion and by the substitution  $t = u^2$ ,  $dt = 2u du$  we finally get

$$\begin{aligned}
 \int_{\mathcal{F}} z dS &= \int_0^{\sqrt{3}} u \cdot \frac{1}{3} (2\sqrt{2} - 1) \cdot (1 + u^2)^{\frac{3}{2}} du \\
 &= \frac{1}{3} (2\sqrt{2} - 1) \int_0^{\sqrt{3}} (1 + u^2)^{\frac{3}{2}} u du \\
 &= \frac{1}{3} (2\sqrt{2} - 1) \int_0^3 (1 + t)^{\frac{3}{2}} \frac{1}{2} dt \\
 &= \frac{1}{3} (2\sqrt{2} - 1) \cdot \frac{1}{2} \left[ \frac{2}{5} (1 + t)^{\frac{5}{2}} \right]_0^3 \\
 &= \frac{1}{15} (2\sqrt{2} - 1) \cdot \left\{ 4^{\frac{5}{2}} - 1 \right\} \\
 &= \frac{31(2\sqrt{2} - 1)}{15}. \quad \diamond
 \end{aligned}$$

### Example 6.3

A. A surface of revolution  $\mathcal{O}$  is obtained by revolving the meridian curve  $\mathcal{M}$  given by

$$r = a(1 + \sin \theta), \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad a > 0,$$

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where  $\theta$  is the angle measured from the  $z$ -axis and  $r$  is the distance to  $(0, 0)$  (an arc of a cardioid, cf. Example 5.2). Find the surface integral

$$I = \int_{\mathcal{O}} \frac{z}{x^2 + y^2 + z^2} dS.$$

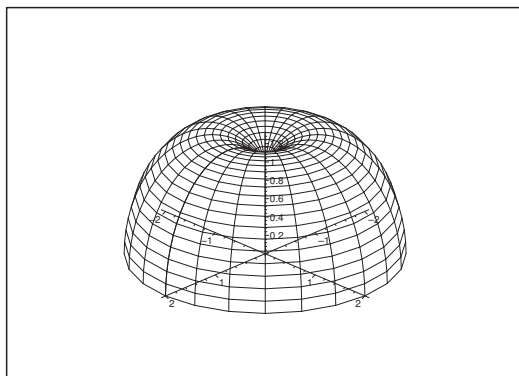


Figure 49: The surface  $\mathcal{O}$  for  $a = 1$ .

An examination of the dimensions shows that  $x, y, z \sim a$  and  $\int_{\mathcal{O}} \dots dS \sim a^2$ , thus

$$\int_{\mathcal{O}} \frac{z}{x^2 + y^2 + z^2} dS \sim \frac{a}{a^2} \cdot a^2 = a.$$

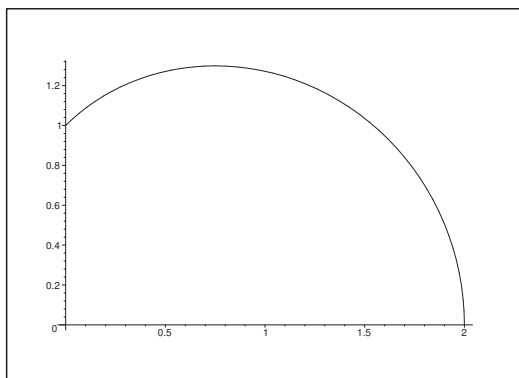


Figure 50: The meridian curve  $\mathcal{M}$  for  $a = 1$ .

The final result must therefore be of the form  $c \cdot a$ , where  $c$  is the constant, we are going to find.

- D.** When we look at surfaces (or bodies) of revolution one should always try either *semi-polar* or *spherical* coordinates. Since the parametric representation of the meridian curve  $\mathcal{M}$  is given in a way which is very similar to the *spherical* coordinates, it is quite reasonable to expect that one should use spherical coordinates.

Although it is here possible to solve the problem by a very nasty trick, it is far better for pedagogical reasons to follow the way which most students would go. Let us analyze the reduction formula

$$\int_{\mathcal{O}} f(x, y, z) dS = \int_a^b \left\{ \int_0^{2\pi} F(t, \varphi) d\varphi \right\} R(t) \sin \Theta(t) \sqrt{\{R'(t)\}^2 + \{R(t)\Theta'(t)\}^2} dt,$$

where

$$F(t, \varphi) := f(R(t) \sin \Theta(t), \cos \varphi, R(t) \sin \Theta(t) \sin \varphi, R(t) \cos \Theta(t)).$$

There is no  $t$  in  $\mathbf{A}$ , so we must start by introducing the parameter  $t$  in a convenient form. Then we shall identify the transformed function  $F(t, \varphi)$  as well as the weight function, and finally we shall carry out all the integrations.

- I. 1) *The introduction of the parameter  $t$ .* The most obvious thing is to by  $\theta = t$ , i.e.  $\mathcal{M}$  is described by

$$r = R(t) = a(1 + \sin t), \quad \theta = \Theta(t) = t, \quad 0 \leq t \leq \frac{\pi}{2}.$$

By doing this we split the different aspects:  $\theta$  belongs to the curve  $\mathcal{M}$ , and  $t$  belongs to the parametric interval

$$\left[0, \frac{\pi}{2}\right] = [a, b].$$

- 2) *Identification of  $F(t, \varphi)$  and the weight function.* Since

$$z = R(t) \cos \Theta(t) = a(1 + \sin t) \cos t \quad \text{on } \mathcal{M},$$

and

$$x^2 + y^2 + z^2 = r^2 = R(t)^2 = a^2(1 + \sin t)^2 \quad \text{on } \mathcal{M},$$

we obtain the *integrand*

$$f(x, y, z) = \frac{z}{x^2 + y^2 + z^2} = \frac{a(1 + \sin t) \cos t}{a^2(1 + \sin t)^2} = \frac{\cos t}{a(1 + \sin t)} = F(t, \varphi),$$

which is independent of  $\varphi$ . Since the weight function and the boundaries of does not depend on  $t$  either, the  $\varphi$ -integral becomes trivial, and we can put  $\int_0^{2\pi} d\varphi = 2\pi$  outside the integral as a factor.

Then we calculate the *weight function*,

$$\begin{aligned} & R(t) \sin \Theta(t) \sqrt{\{R'(t)\}^2 + \{R(t)\Theta'(t)\}^2} \\ &= a(1 + \sin t) \cdot \sin t \cdot \sqrt{\{a \cos t\}^2 + \{a(1 + \sin t) \cdot 1\}^2} \\ &= a(1 + \sin t) \cdot \sin t \cdot a \sqrt{\cos^2 t + (1 + 2 \sin t + \sin^2 t)} \\ &= a^2(1 + \sin t) \cdot \sin t \cdot \sqrt{2(1 + \sin t)} \\ &= \sqrt{2} a^2(1 + \sin t)^{\frac{3}{2}} \cdot \sin t. \end{aligned}$$

3) *Integration by reduction.* First we note that the parametric domain is 2-dimensional,

$$D = \left\{ (\varphi, t) \mid 0 \leq \varphi \leq 2\pi, 0 \leq t \leq \frac{\pi}{2} \right\}.$$

In fact, dimension corresponds to dimension, and since  $\mathcal{F}$  is a  $C^\infty$ -surface, the parametric domain  $D$  must necessarily be 2-dimensional. (If not we have made an error, so start from the very beginning!)

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We have now identified all functions, so we get by the reduction formula that

$$\begin{aligned}
 \int_{\mathcal{O}} \frac{z}{x^2 + y^2 + z^2} dS &= \int_D \frac{\cos t}{a(1 + \sin t)} \cdot \sqrt{2} a^2 (1 + \sin t)^{\frac{3}{2}} \sin t d\varphi dt \\
 &= \sqrt{2} \cdot a \cdot 2\pi \int_0^{\frac{\pi}{2}} \sin t (1 + \sin t)^{\frac{1}{2}} \cos t dt \\
 &= 2\sqrt{2} \pi a \int_0^1 u(1 + u)^{\frac{1}{2}} du \\
 &= 2\sqrt{2} \pi a \int_0^1 (1 + u - 1)(1 + u)^{\frac{1}{2}} du \\
 &= 2\sqrt{2} \pi a \int_0^1 \left\{ (1 + u)^{\frac{3}{2}} - (1 + u)^{\frac{1}{2}} \right\} du \\
 &= 2\sqrt{2} \pi a \left[ \frac{2}{5} (1 + u)^{\frac{5}{2}} - \frac{2}{3} (1 + u)^{\frac{3}{2}} \right]_0^1 \\
 &= 2\sqrt{2} \pi a \cdot \frac{2}{15} \left[ 3(1 + u)^{\frac{5}{2}} - 5(1 + u)^{\frac{3}{2}} \right]_0^1 \\
 &= \frac{4\pi a}{15} \cdot \sqrt{2} \left\{ 3 \left( 2^{\frac{5}{2}} - 1 \right) - 5 \left( 2^{\frac{3}{2}} - 1 \right) \right\} \\
 &= \frac{4\pi a}{15} \sqrt{2} \{ 3(4\sqrt{2} - 1) - 5(2\sqrt{2} - 1) \} \\
 &= \frac{4\pi a}{15} \sqrt{2} \{ 2\sqrt{2} + 2 \} = \frac{8\pi a}{15} \sqrt{2} (\sqrt{2} + 1) \\
 &= \frac{8\pi(2 + \sqrt{2})a}{15}.
 \end{aligned}$$

**C.** *Weak control.* The result has the form  $c \cdot a$ , in agreement with **A**.

Since  $z \geq 0$  on  $\mathcal{O}$  [cf. the figure], the result must be  $\geq 0$ . We see that this is also the case here.  $\diamond$

#### Example 6.4

**A.** A surface of revolution  $\mathcal{O}$  has an arc of a parable  $\mathcal{M}$  as its meridian curve, where this is given by the equation

$$z = \frac{\varrho^2}{a}, \quad 0 \leq \varrho \leq a, \quad a > 0.$$

Find the surface integral

$$I = \int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2 + 4az}} dS.$$



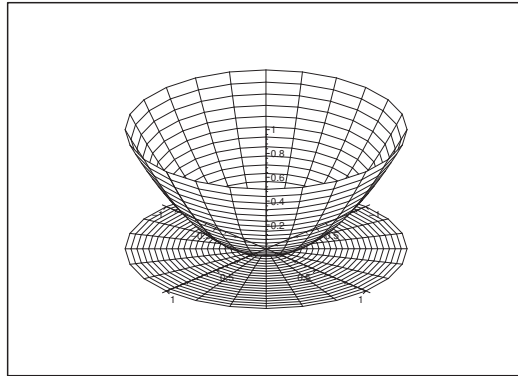


Figure 51: The surface  $\mathcal{O}$  and its projection onto the  $(x, y)$ -plane for  $a = 1$ .

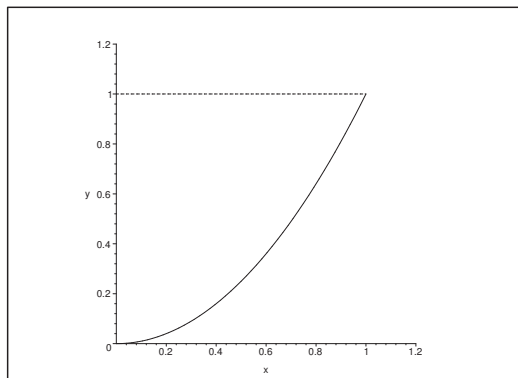


Figure 52: The meridian curve  $\mathcal{M}$  for  $a = 1$ .

*Examination of the dimensions.* It follows from  $x, y, z \sim a$ , that

$$\frac{x^2}{\sqrt{a^2 + 4az}} \sim \frac{a^2}{\sqrt{a^2}} = a.$$

Since  $\int_{\mathcal{O}} \dots dS \sim a^2$ , we get all together

$$\int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2 + 4az}} dS \sim a \cdot a^2 = a^3,$$

i.e. the result must have the form

$$\int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2 + 4az}} dS = c \cdot a^3,$$

where the constant  $c$  must be positive, because the integrand is  $\geq 0$ .

- D.** The description invites to *semi-polar* coordinates **I 1**. For the matter of training we also add **I 2**. *Rectangular coordinates*, which give a slightly different variant, although we in the end is forced back to (semi-)polar coordinates.

I 1. *Semi-polar coordinates.* We introduce  $t$  as a parameter by

$$\rho = P(t) = t.$$

Then

$$z = Z(t) = \frac{1}{a}t^2, \quad 0 \leq t \leq a.$$

Since

$$x = P(t) \cos \varphi = t \cos \varphi, \quad y = P(t) \sin \varphi = t \sin \varphi, \quad z = Z(t) = \frac{1}{a}t^2,$$

we get the following interpretation of the *integrand*,

$$f(x, y, z) = \frac{x^2}{\sqrt{a^2 + 4az}} = \frac{t^2 \cos^2 \varphi}{\sqrt{a^2 + 4t^2}},$$

and the *weight function* is

$$P(t)\sqrt{\{P'(t)\}^2 + \{Z'(t)\}^2} = t\sqrt{1^2 + \left(\frac{2}{a}t\right)^2} = \frac{t}{a}\sqrt{a^2 + 4t^2}.$$

The *parametric domain* is

$$D = \{(t, \varphi) \mid 0 \leq t \leq a, 0 \leq \varphi \leq 2\pi\} = [0, a] \times [0, 2\pi].$$

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Hence we get by a reduction

$$\begin{aligned} \int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2 + 4az}} dS &= \int_D \frac{t^2 \cos^2 \varphi}{\sqrt{a^2 + 4t^2}} \cdot \frac{t}{a} \sqrt{a^2 + 4t^2} dt d\varphi \\ &= \frac{1}{a} \int_D t^3 \cos^2 \varphi dt d\varphi = \frac{1}{a} \int_0^a t^3 dt \cdot \int_0^{2\pi} \cos^2 \varphi d\varphi \\ &= \frac{1}{a} \left[ \frac{1}{4} t^4 \right]_0^a \cdot \int_0^{2\pi} \frac{1 + \cos 2\varphi}{2} d\varphi = \frac{\pi a^3}{4}. \end{aligned}$$

**C 1.** *Weak control.* The result has the right dimension  $[a^3]$ , and it is positive, cf. **A**.

**I 2.** The *rectangular* version. In this case we interpret the surface as the *graph* of the function

$$z = f(x, y) = \frac{1}{a} (x^2 + y^2) \quad \text{for } (x, y) \in E,$$

where the parametric domain is the disc

$$E = \{(x, y) \mid x^2 + y^2 \leq a^2\}.$$

The *weight function* is

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \left(\frac{2x}{a}\right)^2 + \left(\frac{2y}{a}\right)^2} = \frac{1}{a} \sqrt{a^2 + 4x^2 + 4y^2}.$$

We have now found everything which is needed for an application of the reduction theorem:

$$\begin{aligned} \int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2 + 4az}} dS &= \int_E \frac{x^2}{\sqrt{a^2 + 4a \cdot \frac{1}{a}(x^2 + y^2)}} \cdot \frac{1}{a} \sqrt{a^2 + 4x^2 + 4y^2} dx dy \\ &= \frac{1}{a} \int_E \frac{x^2}{\sqrt{a^2 + 4x^2 + 4y^2}} \cdot \sqrt{a^2 + 4x^2 + 4y^2} dx dy = \frac{1}{a} \int_E x^2 dx dy. \end{aligned}$$

From this point it is again most natural to change to *polar coordinates*,

$$\begin{aligned} \int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2 + 4az}} dS &= \frac{1}{a} \int_E x^2 dx dy = \frac{1}{a} \int_0^{2\pi} \left\{ \int_0^a (\varrho \cos \varphi)^2 \cdot \varrho d\varrho \right\} d\varphi \\ &= \frac{1}{a} \int_0^{2\pi} \cos^2 \varphi d\varphi \cdot \int_0^a \varrho^3 d\varrho = \frac{1}{a} \cdot \pi \cdot \frac{a^4}{4} = \frac{\pi a^3}{4}. \quad \diamond \end{aligned}$$

## 7 Transformation theorems

### Example 7.1

A. Calculate the plane integral

$$I = \int_B \cos\left(\frac{y-x}{y+x}\right) dx dy$$

over the trapeze shown on the figure.

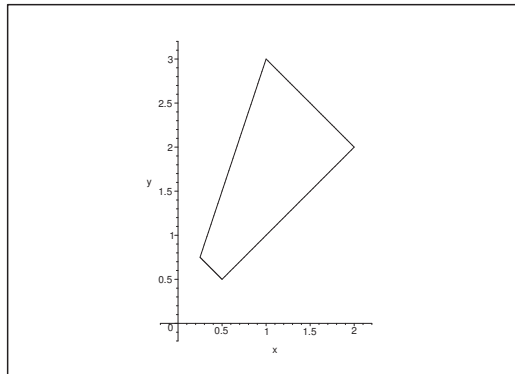


Figure 53: The trapeze  $B$ .

D. A direct calculation applying one of the usual reduction theorems is not possible, because none of the forms

$$\int \cos\left(\frac{y-x}{y+x}\right) dx = \int \cos\left(\frac{2y}{y+x} - 1\right) dx = \int \cos\left(1 - \frac{2x}{y+x}\right) dy$$

can be integrated within the realm of our known functions. The situation is even worse in polar coordinates. Therefore, the only possibility left is to find a convenient transform, such that the integrand becomes more easy to handle.

The unpleasant thing is of course the fraction  $\frac{y-x}{y+x}$ . One idea would be to introduce the numerator as a new variable, and the denominator as another new variable. If we do this, then we must show that we obtain a *unique* correspondence between the domain  $B$  and a *parametric domain*  $D$ , which also should be found. Finally we shall find the *Jacobian*. When we have found all the terms in the transformation formula, then calculate the integral.

**Remark 7.1** This time we see that it is here quite helpful to start the discussion in **D**, which is not common knowledge from high school. First we discuss the problem. Based on this discussion we make a decision on the further procedure.  $\diamond$

I. According to **D**, we choose the numerator and the denominator as our new variables. Most people would here choose the *numerator* as  $u$  and the *denominator* as  $v$ , so we shall do the same, although it can be shown that we shall get simpler calculations if we interchange the definition of  $u$  and  $v$ .

We therefore put as the most natural choice

$$(10) \text{ numerator : } u = y - x \quad \text{and} \quad \text{denominator : } v = y + x.$$

Then we shall prove that this gives a *one-to-one* correspondence. This means that we for any given  $u$  and  $v$  obtain unique solutions  $x$  and  $y$ :

$$x = \frac{v - y}{2} \quad \text{and} \quad y = \frac{u + v}{2}.$$

Obviously the transform is *continuous* both ways. Since  $B$  is *closed and bounded*, the range  $D$  by this transform is again *closed and bounded*, cf. the important *second main theorem for continuous functions*.

Since the transform is one-to-one *everywhere*, the boundary  $\partial B$  is mapped one-to-one onto the boundary  $\partial D$ . This is expressed in the following way:

- 1) The line  $x + y = 1$  corresponds by (10) to  $v = 1$ .
- 2) The line  $y = x$ , i.e.  $y - x = 0$ , corresponds by (10) to  $u = 0$ .
- 3) The line  $y + x = 4$  corresponds by (10) to  $v = 4$ .
- 4) The line  $y = 3x$  corresponds to  $\frac{u + v}{2} = 3 \frac{v - u}{2}$ , i.e. to  $v = 2u$ .

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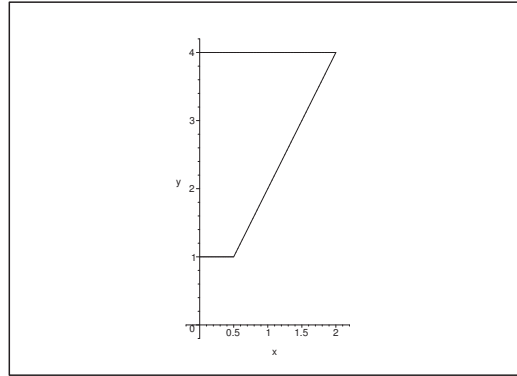


Figure 54: The parametric domain  $D$ . The skew line has the equation  $v = 2u$  or  $u = \frac{1}{2}v$  as its representation.

The only *closed and bounded* domain in the  $(u, v)$ -plane, which has the new boundary curves as its boundary is  $D$  as indicated on the figure. In practice one draws the lines  $v = 1$ ,  $u = 0$ ,  $v = 4$  and  $v = 2u$  and use the figure to find out where the bounded set  $D$  is situated, such that the boundary consists of parts of all four lines.

Then we calculate the *weight function*  $\left| \frac{\partial(x, y)}{\partial(u, v)} \right|$ .

The “old fashioned” notation above indicates that we shall use the transform, where  $x$  and  $y$  (in the “numerator”) are functions of  $u$  and  $v$  (in the “denominator”), i.e.

$$x = \frac{1}{2}v - \frac{1}{2}u \quad \text{and} \quad y = \frac{1}{2}u + \frac{1}{2}v.$$

This gives us the *Jacobian*

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \cdot \frac{1}{2} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = \frac{1}{4} \cdot (-2) = -\frac{1}{2}.$$

It follows that the Jacobian is *negative*, hence the *weight function* becomes

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{2}.$$

**Remark 7.2** If we here interchange  $u$  and  $v$  in (10), then we obtain that the Jacobian becomes *positive*.  $\diamond$

We have now come to the *reduction formula*,

$$I = \int_B \cos\left(\frac{y-x}{y+x}\right) dx dy = \int_D \cos\left(\frac{u}{v}\right) \cdot \frac{1}{2} du dv.$$

Note that both sides here are *abstract plane integrals*.

We see on the right hand side that  $\int \cos\left(\frac{u}{v}\right) dv$  cannot be integrated within the realm of our known arsenal of functions. But  $\int \cos\left(\frac{u}{v}\right) du$  can! Therefore, when we reduce the plane integral on the right hand side we put the  $u$ -integral as the inner integral. Then

$$I = \frac{1}{2} \int_D \cos\left(\frac{u}{v}\right) du dv = \frac{1}{2} \int_1^4 \left\{ \int_0^{\frac{v}{2}} \cos\left(\frac{u}{v}\right) du \right\} dv.$$

When  $v \neq 0$  is kept constant, we get from the inner integral

$$\int_0^{\frac{v}{2}} \cos\left(\frac{u}{v}\right) du = \left[ v \sin\left(\frac{u}{v}\right) \right]_0^{\frac{v}{2}} = v \sin\left(\frac{v}{2} \cdot \frac{1}{v}\right) = \sin\left(\frac{1}{2}\right) \cdot v,$$

where  $\sin\left(\frac{1}{2}\right)$  is a constant, which shall *not* be found explicitly! (Note that at  $\frac{1}{2}$  radian is *not* equal to  $\frac{\pi}{2}$ ).

Finally we get by insertion

$$I = \frac{1}{2} \int_1^4 \sin\left(\frac{1}{2}\right) \cdot v dv = \frac{1}{2} \sin\left(\frac{1}{2}\right) \cdot \left[ \frac{1}{2} v^2 \right]_1^4 = \frac{15}{4} \sin\left(\frac{1}{2}\right). \quad \diamond$$

### Example 7.2

A. Let

$$A = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}, -y \leq z \leq y\}.$$

calculate

$$I = \int_A \frac{\exp((2-y-z)^3)}{4+y+z} d\Omega.$$

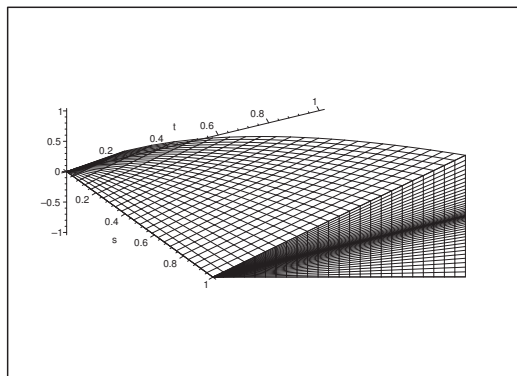


Figure 55: The domain A. Note the different scales on the axes.

D. Let us start by pulling out the teeth of this big and horrible example! Its purpose is *only* to demonstrate that even apparent incalculable integrals in some cases nevertheless can be calculated by using a “convenient transform”. This example is from a textbook, where earlier students got the wrong impression that “every application of the transformation theorem looks like this example”, which is not true. Without this extra comment this example will send a *wrong* message to the reader.

Let us first discuss, how we can find a reasonable transform. I shall follow more or less the way of thinking which the author of this example must have used, the first time it was created. (It would be easier for me to ask him, because I know him, but that is not sporty.)

At the end of this example, in Remark 7.4, I shall describe the very modest requirements which may be demanded of the students. In other words, this example should only be used as an *inspiration* for other similar problems which may occur in practice (but not at the exams).

I. Let us start by looking at the *geometry* of  $A$ . The projection  $B$  of  $A$  onto the  $(x, y)$ -plane is

$$B = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}\}.$$

Since  $A$  for every  $x \in ]0, 1]$  is cut into an isosceles rectangular triangle

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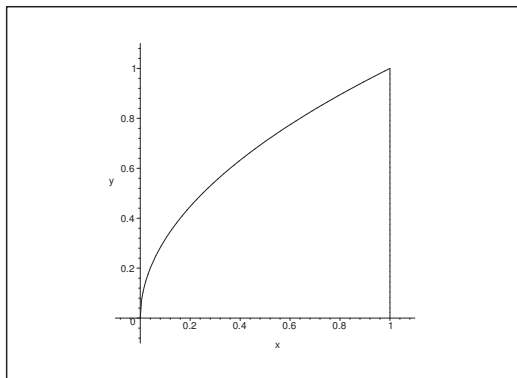


Figure 56: The projection  $B$  of the domain  $A$  onto the  $(x, y)$ -plane.

$$\Delta_x = \{(y, z) \mid 0 \leq y \leq \sqrt{x}, -y \leq z \leq y\},$$

it is easy to sketch  $A$ , cf. a previous figure.

Then the integrand

$$\frac{\exp((2 - y - z)^3)}{4 + y + z}$$

should be “straightened out”. A reasonable guess would be to introduce

$$u = y + z.$$

**Remark 7.3** Once we have gone through all the calculations it can be seen that

$$\tilde{u} = \frac{y + z}{2},$$

would be a better choice, because then we shall get rid of a lot of irritating constants. Of pedagogical reasons we shall *not* here use the most optimal transform, but instead the transform which one would expect the student to choose.  $\diamond$

Since we do not get further information from the integrand, we shall turn to the domain  $A$ . The boundary of  $A$  is (almost) determined by putting equality sign into the definition of  $A$  instead of  $\leq$ . First everything is written in a “binary” way in the definition of  $A$ ,

$$\begin{aligned} A &= \{(x, y, z) \mid 0 \leq x \leq 1 \wedge 0 \leq y \leq \sqrt{x} \wedge -y \leq z \leq y\} \\ &= \{(x, y, z) \mid 0 \leq x \wedge x \leq 1 \wedge 0 \leq y \wedge 0 \leq \sqrt{x} - y \wedge 0 \leq y + z \wedge 0 \leq y - z\}, \end{aligned}$$

i.e. every condition which is defining  $A$  contains only one inequality sign and one of the sides of the inequality is a constant.

We see that there are composed expressions in the latter three conditions,

$$\sqrt{x} - y \geq 0, \quad y + z \geq 0, \quad y - z \geq 0.$$

Since we already have chosen  $u = y + z \geq 0$ , we get the inspiration of choosing the new variables

$$(11) \quad u = y + z \geq 0, \quad v = y - z \geq 0, \quad w = \sqrt{x} - y \geq 0,$$

where we have taken the most ugly term,  $\sqrt{x} - y$  and put it equal to  $w$ , i.e.

$$w = \sqrt{x} - y.$$

We note that we by these choices have obtained that  $u, v, w \geq 0$ , and that equality signs must correspond to boundary points in the  $(u, v, w)$ -space for the parametric domain  $D$ .

Next we show that the transform (11) is one-to-one. i.e. we shall express  $x, y, z$  *uniquely* by  $u, v, w$ . We get immediately from the first two equations that

$$y = \frac{u+v}{2} \quad \text{and} \quad z = \frac{u-v}{2}.$$

From the third equation we get

$$\sqrt{x} = w + y = w + \frac{u+v}{2} = \frac{1}{2}(u+v+2w),$$

which obviously is  $\geq 0$ , because  $u, v, w \geq 0$ . Therefore, by squaring,

$$x = \frac{1}{4}(u+v+2w)^2.$$

Thus,  $x, y, z$  are uniquely determined by  $u, v, w$ , so the transform is *one-to-one*.

Since the transform and its inverse are both *continuous* and the domain  $A$  is *closed and bounded*, it follows from the *second main theorem for continuous functions* that  $D$  is also *closed and bounded*. It follows from the binary representation of  $A$  that  $\partial A$  is a subset of the union of the surfaces  $x = 0, x = 1, y = 0, \sqrt{x} - y = 0, y + z \leq 0$  and  $y - z = 0$ . These are now investigated one by one.

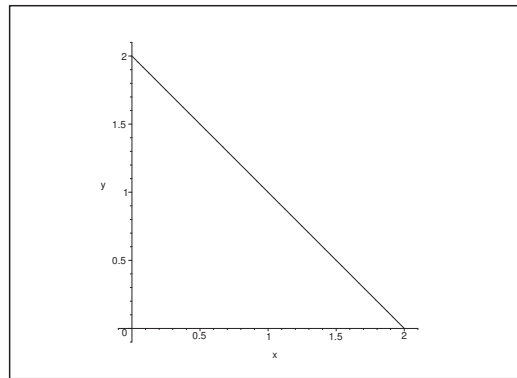


Figure 57: The projection of the parametric domain  $D$  in the  $(u, v)$ -plane.

- 1) The plane  $x = 0$  corresponds to  $\frac{1}{4}(u+v+2w)^2 = 0$ . Since  $u, v, w \geq 0$ , we only get  $(u, v, w) = (0, 0, 0)$ , which is in agreement with the figure of  $A$ , because the plane  $x = 0$  just cuts  $A$  in  $\mathbf{0}$ .
- 2) The plane  $x = 2$  corresponds to  $\frac{1}{4}(u+v+w)^2 = 1$ , i.e.  $\frac{1}{2}(u+v+2w) = +1$ , from which

$$w = 1 - \frac{u+v}{2} \geq 0.$$

Here we have again used that  $u, v, w \geq 0$ . Note that we also get that

$$u + v \leq 2.$$

- 3) The plane  $y = 0$  corresponds to  $\frac{1}{2}(u + v) = 0$ , i.e.  $u + v = 0$ .
- 4) The remaining conditions have been found previously for  $u = 0, v = 0$  and  $w = 0$ .

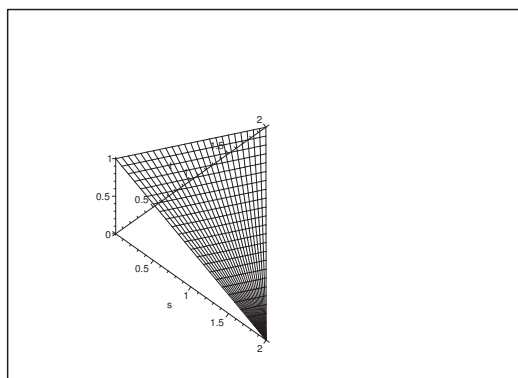


Figure 58: The parametric domain  $D$ .

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Summing up we find that the *parametric domain* is given by

$$\begin{aligned} D &= \left\{ (u, v, w) \mid 0 \leq u, 0 \leq u + v \leq 2, 0 \leq w \leq 1 - \frac{u + v}{2} \right\} \\ &= \left\{ (u, v, w) \mid 0 \leq u \leq 2, 0 \leq v \leq 2 - u, 0 \leq w \leq 1 - \frac{u + v}{2} \right\} \\ &= \left\{ (u, v, w) \mid (u, v) \in B, 0 \leq w \leq 1 - \frac{u + v}{2} \right\}, \end{aligned}$$

where the projection  $B$  in the  $(u, v)$ -plane is given by

$$B = \{(u, v) \mid 0 \leq u \leq 2, 0 \leq v \leq 2 - u\},$$

so  $B$  and  $D$  are now easily sketched.

By the chosen transform the *integrand* is carried over into

$$\frac{\exp((2 - y - z)^3)}{4 + y + z} = \frac{\exp((2 - u)^2)}{4 + u}.$$

Then we calculate the *weight function*  $\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|$ . First note that

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(u, v, w)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(u + v + 2w) & \frac{1}{2}(u + v + 2w) & u + v + 2w \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{vmatrix} \\ &= (u + v + 2w) \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}(u + v + 2w). \end{aligned}$$

Since  $u, v, w \geq 0$  in  $D$ , we see that the *weight function* is

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = +\frac{1}{2}(u + v + 2w).$$

This is only 0 for  $(u, v, w) = (0, 0, 0)$  in  $D$ , i.e. in just one point, which is a null-set (without a positive volume). Therefore we may continue with the transformation theorem in its abstract form:

$$\begin{aligned} I &= \int_A \frac{\exp((2 - y - z)^3)}{4 + y + z} dx dy dz \\ &= \int_D \frac{\exp((2 - u)^3)}{4 + u} \frac{1}{2}(u + v + 2w) du dv dw. \end{aligned}$$

Here it is obvious that we shall *not* start by integrating after  $u$ . If we choose  $u$  as the last (i.e. the outer) variable of integration, then we get by one of the reduction theorems that

$$(12) \quad I = \frac{1}{2} \int_0^2 \frac{\exp((2 - u)^3)}{4 + u} \left\{ \int_{B(u)} (u + v + 2w) dv dw \right\} du,$$

where  $B(u)$  is the cut of  $D$  for  $u$  constant, i.e.

$$B(u) = \left\{ (v, w) \mid 0 \leq v \leq 2 - u, 0 \leq w \leq 1 - \frac{u+v}{2} \right\}.$$

We calculate for fixed  $u \in [0, 2]$  the inner integral in (12) by first integrating vertically with respect to  $w$ :

$$\int_{B(u)} (u + v + 2w) \, dv \, dw = \int_0^{2-u} \left\{ \int_0^{1-\frac{u+v}{2}} (u + v + 2w) \, dw \right\} \, dv.$$

We calculate the inner integral

$$\begin{aligned} \int_0^{1-\frac{u+v}{2}} (u + v + 2w) \, dw &= [(u + v)w + w^2]_0^{1-\frac{u+v}{2}} = [w(u + v + w)]_0^{1-\frac{u+v}{2}} \\ &= \left\{ 1 - \frac{u+v}{2} \right\} \left\{ 1 + \frac{u+v}{2} \right\} = 1 - \frac{1}{4} (u + v)^2. \end{aligned}$$

By insertion we next get for fixed  $u$  that

$$\begin{aligned} \int_{B(u)} (u + v + 2w) \, dw \, dv &= \int_0^{2-u} \left\{ 1 - \frac{1}{4} (u + v)^2 \right\} \, dv = \left[ v - \frac{1}{12} (u + v)^3 \right]_{v=0}^{2-u} \\ &= -(u - 2) + \frac{1}{12} (u^3 - 2^3) \\ &= \frac{1}{12} (u - 2) \{-12 + u^2 + 2u + 4\} \\ &= \frac{1}{12} (u - 2) \{u^2 + 2u - 8\} = \frac{1}{12} (u - 2)^2 (u + 4). \end{aligned}$$

Notice that we here find all factors. When this result is put into (12), we get the reduction

$$\begin{aligned} I &= \frac{1}{2} \int_0^2 \frac{\exp((2-u)^3)}{4+u} \left\{ \int_{B(u)} (u + v + 2w) \, dv \, dw \right\} \, du \\ &= \frac{1}{2} \int_0^2 \frac{\exp((2-u)^3)}{4+u} \cdot \frac{1}{12} (u - 2)^2 (u + 4) \, du \\ &= \frac{1}{24} \int_0^2 \exp((2-u)^3) (u - 2)^2 \, du. \end{aligned}$$

Now, choose the substitution  $t = (2 - u)^3$ . Then  $dt = -3(2 - u)^2 \, du$ , and thus

$$(u - 2)^2 \, du = -\frac{1}{3} \, dt.$$

Finally we get

$$\begin{aligned} I &= \frac{1}{24} \int_0^2 \exp((2-u)^3) (u - 2)^2 \, du = \frac{1}{24} \int_{(2-0)^3}^{(2-2)^3} \exp(t) \cdot \left(-\frac{1}{3}\right) \, dt \\ &= \frac{1}{72} \int_0^8 e^t \, dt = \frac{e^8 - 1}{72}. \end{aligned}$$

**Remark 7.4** It is obvious from this example, that the application of transformation theorems is not an easy job. Therefore, one will always be *given* the transform which should be applied,

$$u = f(x, y, z), \quad v = g(x, y, z), \quad w = h(x, y, z).$$

Then the task for the student can be described in the following points:

- 1) *Solve* the equations after  $x, y, z$ , (from this follows automatically that the transform is *one-to-one*),

$$x = F(u, v, w), \quad y = G(u, v, w), \quad z = H(u, v, w).$$

- 2) *Identify* the parametric domain; use here the second main theorem and that a boundary in most cases by a continuous transform again is mapped into a part of the boundary.

- 3) *Calculate* the weight function  $\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|$  from the expressions found in 1).


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## 8 Improper integrals

### Example 8.1

A. Let  $B = [0, 1]^2$  be the unit square. Examine whether the improper plane integral

$$I = \int_B \frac{1}{y - x - 1} dS$$

is convergent. If this is the case, find its value.

D. The domain  $B$  is closed and bounded; but the integrand is not defined in all points of  $B$ . Cut away open neighbourhoods of the points where the integrand is not defined; notice that the integrand does not change sign; calculate the integral over any of the truncated domains and finally go to the limit.

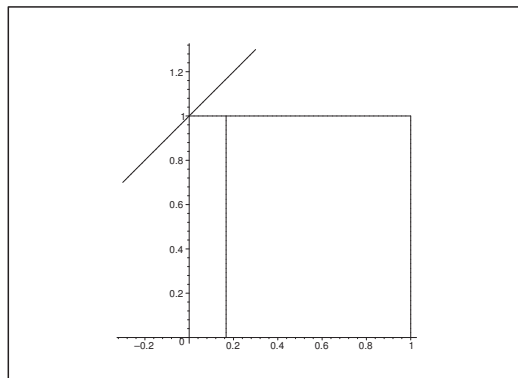


Figure 59: The domain  $B$  with the skew singular line  $y = x + 1$  and the singular point  $(0, 1)$ , and a convenient truncation parallel to the  $Y$ -axis.

I. The denominator must never be zero, so we have to avoid the line  $y = x + 1$ . This line cuts  $B$  at the point  $(0, 1)$ , which must be removed from the domain of integration.

Then we realize that  $\frac{1}{y - x - 1} < 0$  everywhere in  $B \setminus \{(0, 1)\}$ , hence the integrand does not change sign in the part of  $B$ , where it is defined. Therefore we shall not further divide the domain according to the positive and the negative part of the integrand.

Our next step is to truncate  $B$ , such that the singular point  $(0, 1)$  does not lie in any of the closed and bounded domains  $B(\varepsilon)$ , and such that

$$\text{areal}(B \setminus B(\varepsilon)) \rightarrow 0 \quad \text{for } \varepsilon \rightarrow 0.$$

The most reasonable truncations among many possibilities are given by

$$B(\varepsilon) = [\varepsilon, 1] \times [0, 1] \quad \text{and} \quad \tilde{B}(\varepsilon) = [0, 1] \times [\varepsilon, 1],$$

for  $0 < \varepsilon < 1$ . We shall here choose the first one. Then by a reduction,

$$I(\varepsilon) = \int_{B(\varepsilon)} \frac{1}{y - x - 1} dS = \int_{\varepsilon}^1 \left\{ \int_0^1 \frac{1}{y - x - 1} dy \right\} dx.$$

We calculate the inner integral,

$$\int_0^1 \frac{1}{y-x-1} dy = [\ln |y-x-1|]_0^1 = \ln |-x| - \ln |-x-1| = \ln x - \ln(x+1).$$

Then insert the result followed by a partial integration,

$$\begin{aligned} I(\varepsilon) &= \int_{\varepsilon}^1 \{1 \cdot \ln x - 1 \cdot \ln(x+1)\} dx \\ &= [x \ln x]_{\varepsilon}^1 - \int_{\varepsilon}^1 x \cdot \frac{1}{x} dx - [(x+1) \ln(x+1)]_{\varepsilon}^1 + \int_{\varepsilon}^1 (x+1) \cdot \frac{1}{x+1} dx. \end{aligned}$$

The clever trick is here to choose  $\int dx = x$  in the first partial integration, and  $\int dx = x+1$  in the second one. In fact, the antiderivatives are only determined modulo a constant. By this trick we get

$$\begin{aligned} I(\varepsilon) &= 1 \cdot \ln 1 - \varepsilon \ln \varepsilon - 2 \ln 2 + (1+\varepsilon) \ln(1+\varepsilon) \\ &= -2 \ln 2 - \varepsilon \cdot \ln \varepsilon + (1+\varepsilon) \ln(1+\varepsilon). \end{aligned}$$

Since area  $B \setminus B(\varepsilon) = \varepsilon \rightarrow 0$  for  $\varepsilon \rightarrow 0+$ , this is the right limit. It follows from the magnitude of functions that

$$\varepsilon \ln \varepsilon \rightarrow 0 \quad \text{for } \varepsilon \rightarrow 0+,$$

and since

$$(1+\varepsilon) \cdot \ln(1+\varepsilon) \rightarrow 1 \cdot \ln 1 = 0 \quad \text{for } \varepsilon \rightarrow 0+,$$

we conclude that

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0+} I(\varepsilon) \\ &= -2 \ln 2 - \lim_{\varepsilon \rightarrow 0+} \varepsilon \cdot \ln \varepsilon + \lim_{\varepsilon \rightarrow 0+} (1+\varepsilon) \cdot \ln(1+\varepsilon) \\ &= -2 \ln 2, \end{aligned}$$

i.e. the improper integral is *convergent* and its value is

$$I = \int_B \frac{1}{y-x-1} dS = -2 \ln 2 < 0.$$

**C.** *A very weak control.* The integrand is negative, where it is defined. Hence the result should also be negative, which is seen to be true. (Note that this is only catching errors, where we end up with a *positive* result. Negative wrong results cannot be traced in this way).  $\diamond$



**Example 8.2**

A. Let  $\mathcal{F}$  be the half-sphere  $\mathcal{F}$  in the upper half plane of radius  $a > 0$ , i.e. in spherical coordinates,

$$r = a, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq 2\pi.$$

Examine whether the improper surface integral

$$I = \int_{\mathcal{F}} \frac{1}{z} dS$$

is convergent.

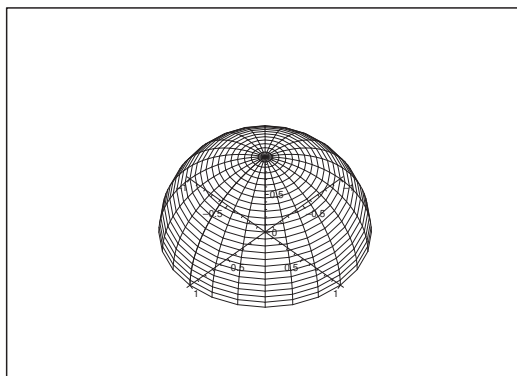


Figure 60: The surface  $\mathcal{F}$  for  $a = 1$ .

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**D.** *Surface integrals* can also be improper!

In the formulation of the task there is a hint of using *spherical* coordinates; though there exist also quite reasonable variants of solutions in *semi-polar* and *rectangular* coordinates.

The points where the integrand is not defined lie on the circle

$$\{(x, y, 0) \mid x^2 + y^2 = a^2\}$$

in the  $(x, y)$ -plane. Hence by the truncations we shall stay away from the  $(x, y)$ -plane. On the residual part of the surface we see that the integrand  $\frac{1}{z} > 0$ , so no further division of the domain is needed concerning the sign of the integrand.

**I 1.** *Spherical coordinates.* In this case we truncate  $\mathcal{F}$  to  $\mathcal{F}(\varepsilon)$  by

$$\mathcal{F}(\varepsilon) : \quad r = a, \quad 0 \leq \theta \leq \frac{\pi}{2} - \varepsilon, \quad 0 \leq \varphi \leq 2\pi, \quad \text{hvor } 0 < \varepsilon < \frac{\pi}{2}.$$

A geometrical consideration gives that

$$\text{area}(\mathcal{F} \setminus \mathcal{F}(\varepsilon)) \rightarrow 0 \quad \text{for } \varepsilon \rightarrow 0+,$$

and as mentioned in **D.** the integrand is positive, so we shall not divide the domain further.

By a reduction of the ordinary *surface integral* over  $\mathcal{F}(\varepsilon)$  we get

$$\begin{aligned} I(\varepsilon) &= \int_{\mathcal{F}(\varepsilon)} \frac{1}{z} dS = \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}-\varepsilon} \frac{1}{a \cos \theta} \cdot a^2 \sin \theta d\theta \right\} d\varphi \\ &= a \cdot 2\pi \int_0^{\frac{\pi}{2}-\varepsilon} \frac{1}{\cos \theta} \cdot \sin \theta d\theta = 2\pi a [-\ln \cos \theta]_0^{\frac{\pi}{2}-\varepsilon} \\ &= 2\pi a \left\{ \ln \cos 0 - \ln \cos \left( \frac{\pi}{2} - \varepsilon \right) \right\} \\ &= -2\pi a \ln \left\{ \cos \frac{\pi}{2} \cdot \cos \varepsilon + \sin \frac{\pi}{2} \cdot \sin \varepsilon \right\} \\ &= -2\pi a \ln \{0 + \sin \varepsilon\} = -2\pi a \ln \sin \varepsilon. \end{aligned}$$

Since  $\sin \varepsilon \rightarrow 0+$  for  $\varepsilon \rightarrow 0+$ , we have  $\ln \sin \varepsilon \rightarrow -\infty$  for  $\varepsilon \rightarrow 0+$ , i.e.  $I(\varepsilon) \rightarrow -(-\infty) = +\infty$  for  $\varepsilon \rightarrow 0+$ , and the improper surface integral is *divergent*.

**I 2.** *Semi-polar coordinates.* Here the surface is described by

$$\mathcal{F} : \quad 0 \leq z \leq a \quad \text{and} \quad \varrho = \sqrt{a^2 - z^2},$$

$$\text{i.e. } \varrho^2 + z^2 = a^2, \quad \varrho \geq 0.$$

The surface is truncated in the following way,

$$\mathcal{F}(\varepsilon) : \quad \varepsilon \leq z \leq a, \quad \varrho = \sqrt{a^2 - z^2}, \quad \text{where } 0 < \varepsilon < a,$$

and it follows geometrically that  $\text{area}(\mathcal{F} \setminus \mathcal{F}(\varepsilon)) \rightarrow 0$  for  $\varepsilon \rightarrow 0+$ .

The surface integral over  $\mathcal{F}(\varepsilon)$  is reduced in the following

$$I(\varepsilon) = \int_{\mathcal{F}(\varepsilon)} \frac{1}{z} dS = \int_{\varepsilon}^a \frac{1}{z} \left\{ \int_{B(z)} ds \right\} dz = \int_{\varepsilon}^a \frac{1}{z} \text{length}\{B(z)\} dz,$$

because  $\mathcal{F}(\varepsilon)$  is cut at height  $z \in [\varepsilon, a]$  in a circle  $B(z)$  of radius  $\varrho = \sqrt{a^2 - z^2}$ , such that the inner integral is the length of  $B(z)$ , i.e.

$$2\pi \varrho = 2\pi \sqrt{a^2 - z^2}.$$

Then we get by insertion

$$I(\varepsilon) = 2\pi \int_{\varepsilon}^a \frac{1}{z} \sqrt{a^2 - z^2} dz.$$

When  $z$  is small, then  $\sqrt{a^2 - z^2} \approx a$ , hence the integrand is  $\approx \frac{a}{z}$ . According to Calculus 1 this function cannot be integrated from 0, so this gives a hint that we may have divergence. Let us prove this.

The integrand is positive everywhere. If  $0 < z \leq \frac{\sqrt{3}}{2}a$ , then

$$\sqrt{a^2 - z^2} \geq \sqrt{a^2 - \frac{3}{4}a^2} = \frac{a}{2}.$$

We then have for  $0 < \varepsilon < \frac{\sqrt{3}}{2}a$  the following estimates

$$\begin{aligned} I(\varepsilon) &= 2\pi \int_{\varepsilon}^a \frac{1}{z} \sqrt{a^2 - z^2} dz \geq 2\pi \int_{\varepsilon}^{\frac{\sqrt{3}}{2}a} \frac{1}{z} \sqrt{a^2 - z^2} dz \\ &\geq 2\pi \int_{\varepsilon}^{\frac{\sqrt{3}}{2}a} \frac{a}{2} \cdot \frac{1}{z} dz = \pi a \int_{\varepsilon}^{\frac{\sqrt{3}}{2}a} \frac{dz}{z} \\ &= \pi a [\ln z]_{\varepsilon}^{\frac{\sqrt{3}}{2}a} = \pi a \left\{ \ln \left( \frac{\sqrt{3}}{2}a \right) - \ln \varepsilon \right\} \rightarrow +\infty \quad \text{for } \varepsilon \rightarrow 0, \end{aligned}$$

and we conclude that the improper surface integral is *divergent*.

**I 3.** *Rectangular coordinates.* In this case we consider  $\mathcal{F}$  as the graph of the function

$$z = f(x, y) = \sqrt{a^2 - x^2 - y^2}, \quad (x, y) \in E,$$

where the parametric domain is

$$E = \{(x, y) \mid x^2 + y^2 \leq a^2\}.$$

The natural truncation of the domain is here

$$E_\varepsilon = \{(x, y) \mid x^2 + y^2 \leq (a - \varepsilon)^2\}, \quad 0 < \varepsilon < a,$$

and it is obvious that  $\text{area}(E \setminus E_\varepsilon) \rightarrow 0$  for  $\varepsilon \rightarrow 0+$ .

The *weight function* is in the case of a graph given by

$$\begin{aligned} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} &= \sqrt{1 + \left(\frac{x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{y}{\sqrt{a^2 - x^2 - y^2}}\right)^2} \\ &= \frac{a}{\sqrt{a^2 - x^2 - y^2}}. \end{aligned}$$

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Then we get by the reduction formula,

$$\begin{aligned}
 I(\varepsilon) &= \int_{\mathcal{F}(\varepsilon)} \frac{1}{z} dS = \int_{E_\varepsilon} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy \\
 &= a \int_{E_\varepsilon} \frac{1}{a^2 - x^2 - y^2} dx dy = a \int_0^{2\pi} \left\{ \int_0^{a-\varepsilon} \frac{1}{a^2 - \varrho^2} \cdot \varrho d\varrho \right\} d\varphi \\
 &= a \cdot 2\pi \int_0^{(a-\varepsilon)^2} \frac{1}{a^2 - t} \cdot \frac{1}{2} dt = \pi a \left[ -\ln |(a^2 - t)| \right]_0^{(a-\varepsilon)^2} \\
 &= \pi a \{ \ln a^2 - \ln \{a^2 - (a - \varepsilon)^2\} \} \\
 &= \pi a \{ 2 \ln a - \ln \{(2a - \varepsilon)\varepsilon\} \} \\
 &= 2\pi a \ln a - \pi a \ln(2a - \varepsilon) - \pi a \ln \varepsilon \\
 &\rightarrow 2\pi a \ln a - \pi a \ln 2a - (-\infty) = +\infty \quad \text{for } \varepsilon \rightarrow 0+,
 \end{aligned}$$

and we conclude again that the improper surface integral is *divergent*.

**Remark 8.1** In this case we could use all the three classical coordinate systems. Note that the three proofs are totally different in their arguments.  $\diamond$

**Example 8.3**

**A.** Let  $B$  be the disc given by  $x^2 + y^2 \leq a^2$ . Find all values of  $\alpha \in \mathbb{R}$ , for which the (proper or improper) plane integral

$$J(\alpha) = \int_B (a^2 - y^2 - x^2)^\alpha dS$$

is convergent. In case of convergence, find its value.

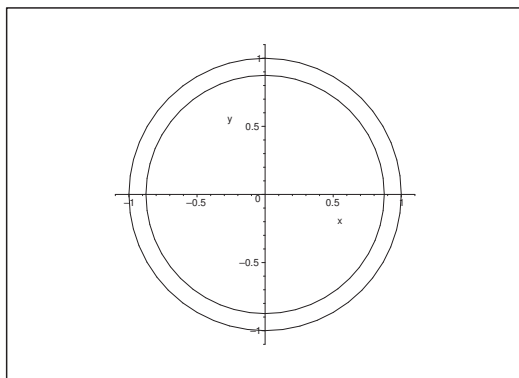


Figure 61: The disc  $B$  and the truncation  $B(p)$  for  $a = 1$ .

This example is of the same type as

$$\int_0^a t^\alpha dt = \begin{cases} \frac{1}{\alpha+1} a^{\alpha+1} & \text{for } \alpha > -1, \\ \text{divergent} & \text{for } \alpha \leq -1, \end{cases}$$

and

$$\int_a^{+\infty} t^\alpha dt = \begin{cases} \text{divergent} & \text{for } \alpha \geq -1, \\ \frac{1}{|\alpha+1|} \cdot \frac{1}{a^{|\alpha+1|}} & \text{for } \alpha < -1, \end{cases}$$

known from Calculus 1.

**D.** *Dimensional considerations* are here extremely useful. In fact,  $x, y \sim a$  and  $\int_B \dots dS \sim a^2$ , hence

$$J(\alpha) \sim a^{2\alpha} \cdot a^2 = a^{2(\alpha+1)}.$$

In order to get convergence we must have

$$a^{2(\alpha+1)} \rightarrow 0 \quad \text{for } a \rightarrow 0+,$$

i.e.  $\alpha + 1 > 1$ , or  $\alpha > -1$ . Since

$$a^{2(\alpha+1)} \rightarrow +\infty \quad \text{for } a \rightarrow 0+, \text{ n\u00e5r } \alpha < -1,$$

we may expect divergence in this case. (Roughly speaking: The integral of a positive term tends to  $+\infty$ , when the domain is shrunk. This is only possible, when we *start* with the value  $+\infty$ , i.e. with divergence).

The integral is proper, when  $\alpha \geq 0$ , and it is improper when  $\alpha < 0$ . When the integrand is defined, it is positive, so we shall not bother with an extra division of the domain according to the positive and negative part of the integrand.

If  $\alpha < 0$ , then the integrand is not defined on the boundary (the circle)  $x^2 + y^2 = a^2$ .

The considerations above indicate that the value  $\alpha = -1$  divides convergence and divergence. For that reason we start by first examining this case.

**I.** When we truncate, it is of no importance whether the integral is proper or improper. Since the integrand is positive (of fixed sign) we shall not make any further division of the domain. We truncate by the definition

$$B(p) = \{(x, y) \mid x^2 + y^2 \leq (pa)^2\}, \quad 0 < p < 1,$$

and we note that  $\text{area}\{B \setminus B(p)\} \rightarrow 0$  for  $p \rightarrow 1-$ .

1) If  $\alpha = -1$ , we get by using *polar* coordinates in  $B(p)$  that

$$\begin{aligned} J(-1; p) &= \int_{B(p)} \frac{1}{a^2 - x^2 - y^2} dS = \int_0^{2\pi} \left\{ \int_0^{pa} \frac{1}{a^2 - \varrho^2} \cdot \varrho d\varrho \right\} d\varphi \\ &= 2\pi \int_0^{(pa)^2} \frac{1}{a^2 - t^2} \cdot \frac{1}{2} dt = \pi \left[ -\ln |a^2 - t| \right]_0^{(pa)^2} \\ &= \pi \{ \ln a^2 - \ln (a^2 - p^2 a^2) \} = \pi \ln \left\{ \frac{a^2}{a^2(1 - p^2)} \right\} \\ &= \pi \ln \left( \frac{1}{1 - p^2} \right) \rightarrow +\infty \quad \text{for } p \rightarrow -1. \end{aligned}$$

We therefore conclude that

$$J(-1) = \int_B \frac{dS}{a^2 - x^2 - y^2} = \lim_{p \rightarrow 1^-} J(-1; p) = +\infty,$$

is divergent.

2) If  $\alpha < -1$ , i.e.  $\alpha + 1 < 0$ , then we can use 1) in the following rearrangements and estimates,

$$\begin{aligned} (a^2 - x^2 - y^2)^\alpha &= (a^2 - x^2 - y^2)^{\alpha+1} \cdot \frac{1}{a^2 - x^2 - y^2} \\ &= a^{2(\alpha+1)} \cdot \left\{ 1 - \frac{x^2 + y^2}{a^2} \right\}^{\alpha+1} \cdot \frac{1}{a^2 - x^2 - y^2} \\ &= a^{2(\alpha+1)} \cdot \frac{1}{\left\{ 1 - \frac{x^2 + y^2}{a^2} \right\}^{|\alpha+1|}} \cdot \frac{1}{a^2 - x^2 - y^2} \\ &\geq a^{2(\alpha+1)} \cdot \frac{1}{a^2 - x^2 - y^2}, \end{aligned}$$

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because

$$\left\{1 - \frac{x^2 + y^2}{a^2}\right\}^{|\alpha+1|} \leq 1.$$

Then it follows from 1) that

$$\begin{aligned} J(\alpha; p) &= \int_{B(p)} (a^2 - x^2 - y^2)^\alpha dS \geq a^{2(\alpha+1)} \int_{B(p)} \frac{1}{a^2 - x^2 - y^2} dS \\ &= a^{2(\alpha+1)} J(-1; p) \rightarrow +\infty \quad \text{for } p \rightarrow 1-, \end{aligned}$$

i.e.  $J(\alpha; p) \rightarrow +\infty$  for  $p \rightarrow 1-$ , and we have got *divergence* for  $\alpha < -1$ , and thus all together for  $\alpha \leq -1$ .

3) If  $\alpha > -1$ , i.e.  $\alpha + 1 > 0$ , we get by using *polar* coordinates in  $B(p)$  that

$$\begin{aligned} J(\alpha; p) &= \int_{B(p)} (a^2 - x^2 - y^2)^\alpha dS = \int_0^{2\pi} \int_0^{p\alpha} (a^2 - \varrho^2)^\alpha \cdot \varrho d\varrho d\varphi \\ &= 2\pi \int_{a^2}^{a^2(1-p^2)} t^\alpha \cdot \left(-\frac{1}{2}\right) dt = \pi \left[ -\frac{1}{\alpha+1} t^{\alpha+1} \right]_{a^2}^{a^2(1-p^2)} \\ &= \frac{\pi}{\alpha} \left\{ a^{2(\alpha+1)} - a^{2(\alpha+1)} \cdot (1-p^2)^{\alpha+1} \right\} \\ &= \frac{\pi}{\alpha+1} a^{2(\alpha+1)} \left\{ 1 - (1-p^2)^{\alpha+1} \right\}. \end{aligned}$$

Since  $\alpha + 1 > 0$  and  $1 - p^2 \rightarrow 0+$  for  $p \rightarrow 1-$ , it follows that

$$(1 - p^2)^{\alpha+1} \rightarrow 0 \quad \text{for } p \rightarrow 1-.$$

Hence the integral is *convergent* for  $\alpha > -1$  with the value

$$J(\alpha) = \int_B (a^2 - x^2 - y^2)^\alpha dS = \frac{\pi}{\alpha+1} a^{2(\alpha+1)}.$$

Summing up we have proved that

$$J(\alpha) = \int_B (a^2 - x^2 - y^2)^\alpha dS = \begin{cases} \frac{\pi}{\alpha+1} a^{2(\alpha+1)} & \text{for } \alpha > -1; \quad \text{convergence;} \\ \infty & \text{for } \alpha \leq -1; \quad \text{divergence.} \end{cases}$$

When the integrand has fixed sign, we allow ourselves to put the value equal to  $+\infty$  (positive integrand) or  $-\infty$  (negative integrand).

However, we shall *not* allow this notation, if both the positive part and the negative part are infinite, because  $\infty - \infty$  does not make sense. We shall return to this in Example 8.5.  $\diamond$

### Example 8.4

**A.** In advanced technical literature one often sees the improper 1-dimensional integral

$$\int_0^\infty e^{-x^2} dx.$$

Obvious applications can be found in Probability and Statistics (the normal distribution); but one can also find it in many other places (the heat equation, diffusion). Find the value of this important integral.



D. The integral cannot be calculated by methods from Calculus 1. It is fairly easy to prove that it is convergent. In fact, if we introduce the function

$$f(t) = (1+t)e^{-t} \quad \text{with } f'(t) = -te^{-t} < 0 \quad \text{for } t > 0,$$

it follows that because  $f(t)$  is decreasing for  $t > 0$  we have

$$(1+t)e^{-t} \leq f(0) = 1, \quad \text{i.e. } e^{-t} \leq \frac{1}{1+t} \quad \text{for } t \geq 0.$$

If we put  $t = x^2$ , we get

$$e^{-x^2} \leq \frac{1}{1+x^2},$$

hence

$$0 < \int_0^n e^{-x^2} dx \leq \int_0^n \frac{dx}{1+x^2} = \text{Arctan } n \rightarrow \frac{\pi}{2} \quad \text{for } n \rightarrow +\infty.$$

This proves the convergence, and even the estimate

$$(0 \leq) \int_0^\infty e^{-x^2} dx \leq \frac{\pi}{2}.$$

However, we still have not found the exact value.

It is now possible to find the value by using methods from Calculus 2. The trick is to consider the improper plane integral

$$I = \int_B \exp(-x^2 - y^2) dS, \quad \text{where } B = [0, +\infty]^2 \text{ is the first quadrant.}$$

The integrand is defined and positive everywhere, so we shall not make any further division of the domain according to the sign of the integrand.

The domain of integration is unbounded, so we must truncate it in a bounded way. We have two obvious possibilities of doing this, depending on whether we consider *polar* or *rectangular* coordinates. The idea is to use both of them, because we by using the *polar* coordinates obtain the *value* of the integral  $I$ , and by using the *rectangular* coordinates we obtain the connection to the integral under consideration.

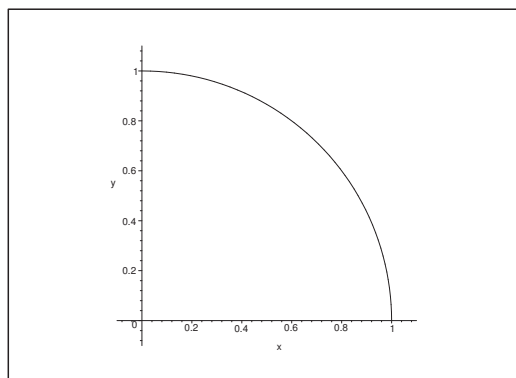


Figure 62: The domain  $Q(R)$  for  $R = 1$ .

1) *Polar coordinates.* We truncate by taking the intersection of  $B$  and discs of radius  $R$ ,

$$Q(R) = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq R^2\}.$$

We reach every point in the first quadrant  $B$  by taking the limit  $R \rightarrow +\infty$ .

When we apply the reduction theorem in polar coordinates over  $Q(R)$  we get

$$\begin{aligned} I(R) &= \int_{Q(R)} \exp(-x^2 - y^2) \, dS = \int_0^{\frac{\pi}{2}} \left\{ \int_0^R \exp(-\varrho^2) \, \varrho \, d\varrho \right\} d\varphi \\ &= \frac{\pi}{2} \int_0^{R^2} e^{-t} \cdot \frac{1}{2} dt = \frac{\pi}{4} [e^{-t}]_0^{R^2} = \frac{\pi}{4} \{1 - \exp(-R^2)\}. \end{aligned}$$

Since  $\exp(-R^2) \rightarrow 0$  for  $R \rightarrow +\infty$ , we conclude that the improper integral is *convergent* with the value

$$I = \int_B \exp(-x^2 - y^2) \, dS = \lim_{R \rightarrow \infty} I(R) = \frac{\pi}{4}.$$

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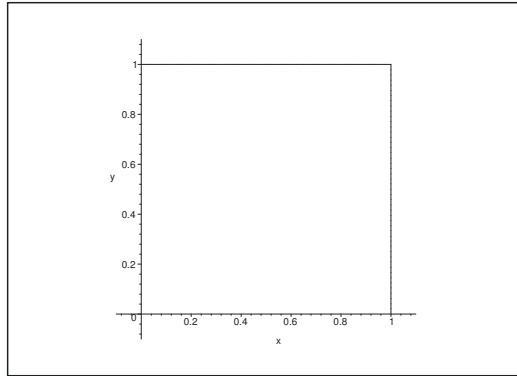
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Figure 63: The domain  $R(n)$  for  $n = 1$ .

2) *Rectangular coordinates.* The truncation is here

$$R(n) = \{(x, y) \mid 0 \leq x \leq n, 0 \leq y \leq n\} = [0, n]^2.$$

We get every point in the first quadrant  $B$  by taking the limit  $n \rightarrow +\infty$ .

When we apply the rectangular reduction theorem over  $R(n)$  we get

$$\begin{aligned} J(n) &= \int_{R(n)} \exp(-x^2 - y^2) dS = \int_0^n \left\{ \int_0^n e^{-x^2} \cdot e^{-y^2} dx \right\} dy \\ &= \int_0^n e^{-x^2} dx \cdot \int_0^n e^{-y^2} dy = \left\{ \int_0^n e^{-t^2} dt \right\}^2. \end{aligned}$$

3) *Summary.* According to 1) the improper integral is convergent, hence the limit can be taken in 2). Since the limit is the same, no matter which truncation we are using, we must have

$$I = \frac{\pi}{4} = \int_B \exp(-x^2 - y^2) dS = \lim_{n \rightarrow \infty} J(n) = \left\{ \int_0^\infty e^{-t^2} dt \right\}^2.$$

Since  $\int_0^\infty e^{-t^2} dt > 0$ , we finally get the value of the integral

$$\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}.$$

**Remark 8.2** It follows that

$$I = \frac{\sqrt{\pi}}{2} \leq \frac{\pi}{2},$$

cf. a remark in section **D**.  $\diamond$

**Example 8.5**

A. Examine whether the improper plane integral

$$\int_{\mathbb{R}^2} \frac{xy}{(1+x^2)(1+y^2)} dS$$

is convergent or divergent.

D. An analysis of the sign shows that the integrand is positive in the first and the third quadrant, while it is negative in the second and the fourth quadrant, so we shall also divide the domain according to the sign. Here we shall also demonstrate the *wrong* argument where one forgets to divide according to the sign of the integrand. This shows that one has to be careful here.

I. We see that the integrand changes its sign if we reflect it in either the  $x$ -axis or in the  $y$ -axis. This shows that the integral over *any* bounded set  $B$ , which is symmetric with respect to at least one of the axes, is 0,

$$\int_B \frac{xy}{(1+x^2)(1+y^2)} dS = 0.$$

The usual truncations in Calculus 2 (discs or centred squares) satisfy this symmetry with respect to both axes, so if one is not too careful one will *erroneously* conclude in the limit that the plane integral is “convergent with the value  $\lim_{n \rightarrow \infty} 0 = 0$ ”.

We shall not prove that this conclusion is *not* correct. We choose this time the truncation

$$B(n) = [0, n] \times [0, n],$$

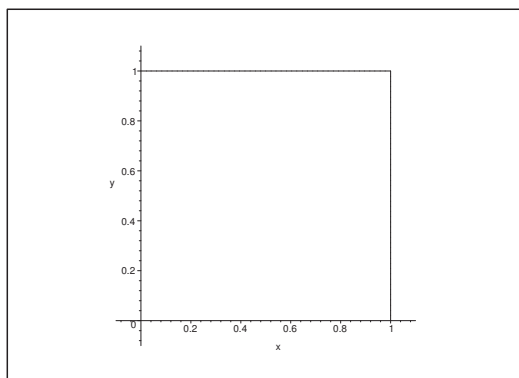


Figure 64: The truncation  $B(n)$  for  $n = 1$ .

i.e. we only consider a subset of the set, where the integrand is  $\geq 0$ . If something goes wrong here in the first quadrant, then it also is wrong in any bigger subset of  $[0, n]^2 \cup [-n, 0]^2$ .

By reduction over  $B(n)$  we get

$$\begin{aligned} \int_{B(n)} \frac{xy}{(1+x^2)(1+y^2)} dS &= \int_0^n \left\{ \int_0^n \frac{xy}{(1+x^2)(1+y^2)} dx \right\} dy \\ &= \int_0^n \frac{x}{1+x^2} dx \cdot \int_0^n \frac{y}{1+y^2} dy = \left\{ \int_0^n \frac{t}{1+t^2} dt \right\}^2 \\ &= \left\{ \left[ \frac{1}{2} \ln(1+x^2) \right]_0^n \right\}^2 = \frac{1}{4} \{ \ln(1+n^2) \}^2 \\ &\rightarrow +\infty \quad \text{for } n \rightarrow \infty. \end{aligned}$$

This shows that the improper integral is *divergent*, thus our first argument must be wrong! (Of the type  $\infty - \infty$ ).

**Remark 8.3** We shall demonstrate how wrong this illegal method is. If we choose the “skew” truncation

$$Q(a, n) = [-an, n] \times [-an, n], \quad a > 0 \text{ constant,}$$

we still get  $\mathbb{R}^2$  by taking the limit  $n \rightarrow +\infty$ . We get by a rectangular reduction,

$$\begin{aligned} \int_{Q(a,n)} \frac{xy}{(1+x^2)(1+y^2)} dS &= \left\{ \int_{-an}^n \frac{t}{1+t^2} dt \right\}^2 = \left\{ \left[ \frac{1}{2} \ln(1+t^2) \right]_{-an}^n \right\}^2 \\ &= \left\{ \frac{1}{2} \ln \left( \frac{1+n^2}{1+a^2n^2} \right) \right\}^2 = \left\{ \frac{1}{2} \ln \left( \frac{1+\frac{1}{n^2}}{a^2+\frac{1}{n^2}} \right) \right\}^2 \\ &\rightarrow \{\ln a\}^2 \quad \text{for } n \rightarrow \infty. \end{aligned}$$

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Every calculation is correct. The only thing which is *wrong* is that the assumptions of taking the limit (with respect to the conclusion of convergence) are *not* satisfied. We note that  $\{\ln a\}^2$  go through the whole interval  $[0, +\infty[$ , when  $a$  go through  $\mathbb{R}_+$ , which means that we can obtain any  $q \geq 0$  as a candidate for a limit of the improper plane integral, which is nonsense.

If we instead use the truncations

$$R(a, n) = [-n, an] \times [-an, n], \quad a > 0 \text{ konstant,}$$

we obtain analogously all negative numbers as possible limits. But *if* the limit exists, then it is unique! Hence the improper plane integral is *divergent*.  $\diamond$

## 9 Vector analysis

### 9.1 Tangential line integral; gradient field

#### Example 9.1

A. Given the vector field

$$\mathbf{V}(x, y, z) = (y^2 + z, 2xy + 2yz^2, 2y^2z + x), \quad (x, y, z) \in \mathbb{R}^3.$$

Examine whether  $\mathbf{V}(x, y, z)$  has an antiderivative. If this is the case, indicate all antiderivatives.

D. Problems of this type can be solved in many ways. One may assume that

1) *No antiderivative exists.* In this case we have two possibilities of proving this claim:

a) Find a pair of indices  $(i, j)$ , such that

$$\frac{\partial V_i}{\partial x_j} \neq \frac{\partial V_j}{\partial x_i}.$$

If we can do this, then the *necessary* condition for an antiderivative is not fulfilled, so there is no antiderivative.

b) Find a *closed* curve  $\mathcal{C}$ , such that the circulation

$$\oint_{\mathcal{C}} \mathbf{V} \cdot d\mathbf{x} \neq 0.$$

Then it follows from the theorem of circulation that no antiderivative exists.

2) *An antiderivative exists.* Here we have got four methods:

a) *Inspection.* This method is very elegant, but one should have a lot of experience before one can rely on it. A final check of the solution is recommended, though it is not mandatory.

b) *Indefinite integration.* In this case the check of the solution is build into the method.

c) *Integration a curve consisting of axis parallel lines.* **The standard method.** It has the drawback that one must *not* forget to *check* the solution afterwards.

d) *Radial integration.* This method is very difficult to perform, because it invites to errors of calculation! The reader is strongly dissuaded to use it for the same reason. Also in this case the final check is mandatory.

In the explicit example we first believe that *antiderivatives exist*. We shall only apply the first three of the possible methods of solution.

I. We shall in all variants start by calculating

$$(13) \quad \mathbf{V} \cdot d\mathbf{x} = (y^2 + z) dx + (2xy + 2yz^2) dy + (2y^2z + x) dz.$$

I 1. *Inspection.*

Calculate (13) in all details and then start to put as much as possible under the  $d$ -sign by using the rule of calculation

$$F'(t) dt = dF(t).$$

Then one should pair all terms which look alike and then use some other of the well-known rules of calculations for differentials, like e.g.

$$f dg + g df = d(fg) \quad \text{and} \quad \frac{1}{g} df - \frac{f}{g^2} dg = d\left(\frac{f}{g}\right).$$

If we in this way can rewrite (13) into

$$\mathbf{V} \cdot d\mathbf{x} = dF(x, y, z) = \nabla F \cdot d\mathbf{x},$$

then  $F(x, y, z)$  is an antiderivative of  $\mathbf{V}$ , and  $\mathbf{V}$  is a gradient field,  $\mathbf{V} = \nabla F$ .

In the specific case we get

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= y^2 dx + z dx + 2xy dy + 2yz^2 dy + 2y^2 z dz + x dz \\ &= y^2 dx + z dx + x d(y^2) + z^2 d(y^2) + y^2 d(z^2) + x dz \\ &= \{z dx + x dz\} + \{y^2 dx + x d(y^2)\} + \{z^2 d(y^2) + y^2 d(z^2)\} \\ &= d(xz) + d(xy^2) + d(y^2 z^2) \\ &= d(xy^2 + xz + y^2 z^2) = dF(x, y, z). \end{aligned}$$

Since we have put everything under the  $d$ -sign, we conclude that  $\mathbf{V}$  is a gradient field,  $\mathbf{V} = \nabla F$ , and that the set of all antiderivatives is given by

$$F(x, y, z) = xy^2 + xz + y^2 z^2 + c,$$

where  $c \in \mathbb{R}$  is an arbitrary constant.

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**I 2. Indefinite integration.** Choose the simplest of the terms in (13). Here we pick up

$$(y^2 + z) dx.$$

(Note that one *must only* choose *one* of the differentials  $dx$ ,  $dy$ ,  $dz$ ).

By an indefinite integration of this we find a *candidate* for an antiderivative:

$$F_1(x, y, z) = \int (y^2 + z) dx = (y^2 + z)x.$$

Then check, whether we have already hit the bull's eye:

$$dF_1 = \frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz = (y^2 + z) dx + 2yx dy + x dz.$$

We did not do it this time, but we see that

$$\mathbf{V} \cdot d\mathbf{x} - dF_1 = 2yz^2 dy + 2y^2 z dz,$$

where  $x$  (which was the variable of integration in the construction of  $F_1(x, y, z)$ ) *has totally disappeared from the right hand side*. If this is *not* the case, we only have two possibilities:

- 1) We have made an error in our calculations!
- 2) There is no error in our calculations, so we conclude that there exists *no* antiderivative, and  $\mathbf{V}$  is not a gradient field.

If one ends up in this situation, one should obviously check the calculations carefully, before one jumps to any conclusion.

Choose one of the terms in the reduced form  $\mathbf{V} \cdot d\mathbf{x} - dF_1$ , and integrate e.g.

$$F_2(y, z) = \int 2yz^2 dy = z^2 \int 2y dy = y^2 z^2.$$

Then check the solution once more:

$$\begin{aligned} dF_2 &= \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz \\ &= 2yz^2 dy + 2y^2 z dz = \mathbf{V} \cdot d\mathbf{x} - dF_1. \end{aligned}$$

This time we hit the bull's eye! Thus we get by a rearrangement

$$\mathbf{V} \cdot d\mathbf{x} = dF_1 + dF_2 = d(xy^2 + xz + y^2 z^2).$$

All antiderivatives are therefore given in the form

$$F(x, y, z) = xy^2 + xz + y^2 z^2 + c,$$

where  $c$  is an arbitrary constant.

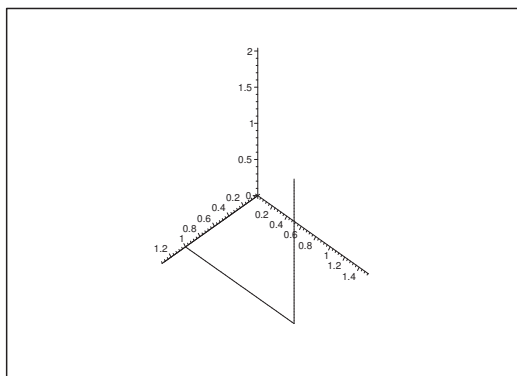


Figure 65: An example of a curve consisting of axis parallel lines, starting from  $(0, 0, 0)$ .

**I 3.** *Integration along a curve consisting of axis parallel lines.* Whenever possible one integrates along the curve defined by

$$(0, 0, 0) \longrightarrow (x, 0, 0) \longrightarrow (x, y, 0) \longrightarrow (x, y, z),$$

with some necessary modifications, if this curve does not lie inside the domain of the integrand.

*Parametric representation:*

$$\begin{aligned} \Gamma_1 : \mathbf{r}_1(t) &= (t, 0, 0), & d\mathbf{x} &= (dt, 0, 0), & t & \text{between } 0 \text{ and } x, \\ \Gamma_2 : \mathbf{r}_2(t) &= (x, t), & d\mathbf{x} &= (0, dt, 0), & t & \text{between } 0 \text{ and } y, \\ \Gamma_3 : \mathbf{r}_3(t) &= (x, y, t), & d\mathbf{x} &= (0, 0, dt), & t & \text{between } 0 \text{ and } z. \end{aligned}$$

Note that we write “ $t$  between 0 and  $x$ ”, and not “ $t \in [0, x]$ ”, because the latter is only correct when  $x \geq 0$ . If instead  $x < 0$ , then  $t \in [x, 0]$ . Analogously in the other cases.

From (13), i.e.

$$\mathbf{V} \cdot d\mathbf{x} = (y^2 + z) dx + (2xy + 2yz^2) dy + (2y^2z + x) dz,$$

we get the line integral, and thus a *candidate*,

$$\begin{aligned} F(x, y, z) &= \int_{\Gamma_1} \mathbf{V} \cdot d\mathbf{x} + \int_{\Gamma_2} \mathbf{V} \cdot d\mathbf{x} + \int_{\Gamma_3} \mathbf{V} \cdot d\mathbf{x} \\ &= \int_0^x (0^2 + 0^2) dt + \int_0^y (2xt + 2t \cdot 0^2) dt + \int_0^z (2y^2t + x) dt \\ &= 0 + xy^2 + y^2z^2 + xz. \end{aligned}$$

Therefore a *candidate* of an antiderivative is

$$F(x, y, z) = xy^2 + xz + y^2z^2.$$

This does not mean that we hereby already have proved that  $F(x, y, z)$  is an antiderivative! The *check* of the solution is here *mandatory*, i.e. we shall check the equation  $\nabla F = \mathbf{V}$ . In this case under consideration we get

$$\begin{aligned}\nabla F &= \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) \\ &= (y^2 + z, 2xy + 2yz^2, 2y^2z + x) = \mathbf{V}(x, y, z).\end{aligned}$$

Thus we have *proved*, that  $F$  is an antiderivative.

Then we get all possible antiderivatives by adding an arbitrary constant  $c \in \mathbb{R}$ :

$$F(x, y, z) = xy^2 + xz + y^2z^2 + c.$$

**Remark 9.1** The check is built into the first two methods, while the third does *not* contain this test. The problem is that an integration along a curve of the described type will *always* produce a candidate, even in cases where  $\mathbf{V}$  is *not* a gradient field. This failure of the method can only be found by a mandatory check.

Note that if one *always* checks the result no matter the choice of method, then it is *waste of time* also to check the additional necessary conditions,

$$\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i}, \quad \text{for every pair of indices } (i, j).$$

One can simply do without this often quite long examination.  $\diamond$

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**Example 9.2**

**A** Given the vector field

$$\mathbf{V}(x, y) = (V_x(x, y), V_y(x, y)) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right), \quad (x, y) \neq (0, 0).$$

Examine whether  $\mathbf{V}(x, y)$  is a gradient field. If this is the case, find all antiderivatives.

**D** The domain  $\mathbb{R}^2 \setminus \{(0, 0)\}$  has a “hole”, thus it is *not* simply connected, and the usual *sufficient* condition is *not* fulfilled. We guess that *there exists an antiderivative*. We shall try all three recommended methods (and we leave out again the radial integration).

**I 1. Inspection.** By using the rules of calculations we get

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy = \frac{1}{\sqrt{x^2 + y^2}} \{x dx + y dy\} \\ &= \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2}} d(x^2 + y^2) = d\left(\sqrt{x^2 + y^2}\right), \end{aligned}$$

from which we conclude that  $\mathbf{V}$  is a gradient field and that all antiderivatives for  $\mathbf{V}$  are given by

$$F(x, y) = \sqrt{x^2 + y^2} + c, \quad (x, y) \neq (0, 0), \quad c \text{ arbitrary constant.}$$

**I 2. Indefinite integration.** If we choose the first term in  $\mathbf{V} \cdot d\mathbf{x}$  as our integrand, we find a *candidate*

$$F_1(x, y) = \int \frac{x}{\sqrt{x^2 + y^2}} dx = \sqrt{x^2 + y^2}, \quad \text{for } (x, y) \neq (0, 0).$$

By the *check* we get

$$\nabla F_1 = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) = \mathbf{V},$$

so even our first guess was correct.

Hence,  $F_1(x, y) = \sqrt{x^2 + y^2}$  is an antiderivative, and  $\mathbf{V}$  is a gradient field.

Every antiderivative is of the form

$$F(x, y) = \sqrt{x^2 + y^2} + c, \quad (x, y) \neq (0, 0), \quad c \text{ arbitrary constant.}$$

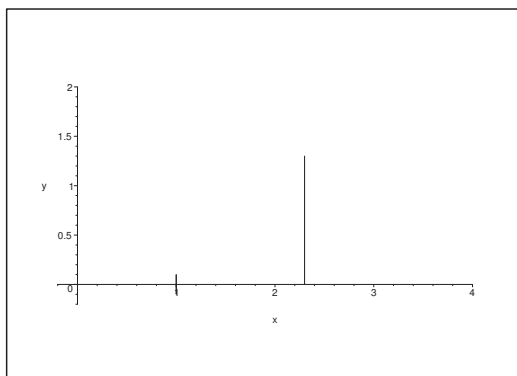


Figure 66: The broken line of integration from  $(1, 0)$  to  $(x, y)$  in the open right half-plane.

**I 3. Integration along a broken line.** Since  $(0, 0)$  does *not* lie in the domain, we choose instead as the path of integration

$$(1, 0) \longrightarrow (x, 0) \longrightarrow (x, y).$$

We see that every point in the *right* half-plane  $x > 0$  can be reached in this way, while no point in the left half-plane  $x \leq 0$  can be reached in this way. Therefore, we are formally only working in the right half-plane. There are apparently problems here, but we shall nevertheless continue our calculations in order to see if we can obtain a reasonable guess. At least we can find the *candidate* for  $x > 0$ ,

$$\begin{aligned} F_1(x, y) &= \int_1^x V_x(t, 0) dt + \int_0^y V_y(x, t) dt \\ &= \int_1^x \frac{t}{\sqrt{t^2 + 0^2}} dt + \int_0^y \frac{t}{\sqrt{x^2 + t^2}} dt \\ &= [t]_1^x + \left[ \sqrt{x^2 + t^2} \right]_0^y = (x - 1) + \left( \sqrt{x^2 + y^2} - \sqrt{x^2 + 0^2} \right) \\ &= \sqrt{x^2 + y^2} - 1, \quad \text{because } \sqrt{x^2} = x \text{ for } x > 0. \end{aligned}$$

Even if the candidate was only derived in the half-plane  $x > 0$ , we see that this *candidate*  $F_1(x, y)$  is of class  $C^1$  in  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , so the following mandatory check is *valid in the whole domain*:

$$\nabla F_1 = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) = \mathbf{V}(x, y), \quad \text{for } (x, y) \neq (0, 0).$$

Let us recapitulate this very *important* argument: The candidate is only found in a subset of the domain, but it is easily seen that the expression is defined in the whole of the *domain!* By checking the candidate we prove that we have thus found a solution in the whole domain, and no further calculation is needed.

We get every antiderivative by adding an arbitrary constant,

$$F(x, y) = \sqrt{x^2 + y^2} + c, \quad (x, y) \neq (0, 0), \quad c \text{ arbitrary constant.} \quad \diamond$$

## 9.2 Flux and divergence of a vector field; Gauß's theorem

### Example 9.3

A Find the flux  $\Phi_1$  of the vector field

$$\mathbf{V}(x, y, z) = (yz, -xz, x^2 + y^2), \quad (x, y, z) \in \mathbb{R}^3,$$

through the surface  $\mathcal{F}$  given by

$$\mathbf{r}(u, v) = (u \sin v, u \cos v, uv), \quad 0 \leq u \leq 1, \quad 0 \leq v \leq u.$$

D By using the reduction theorem

$$\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS = \int_E \mathbf{V}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v) \, du \, dv,$$

where an abstract *surface* integral is reduced to an abstract *plane* integral, it follows that we shall

- 1) identify the parametric domain  $E$ ,
- 2) find the normal vector  $\mathbf{N}(u, v)$  for the *surface*  $\mathcal{F}$  in the given parametric representation,
- 3) find  $\mathbf{V}(\mathbf{r}(u, v))$  on the surface as a function in the new variables  $(u, v)$ .

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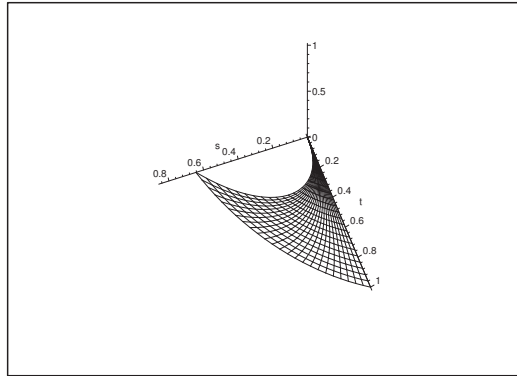


Figure 67: The surface  $\mathcal{F}$ .

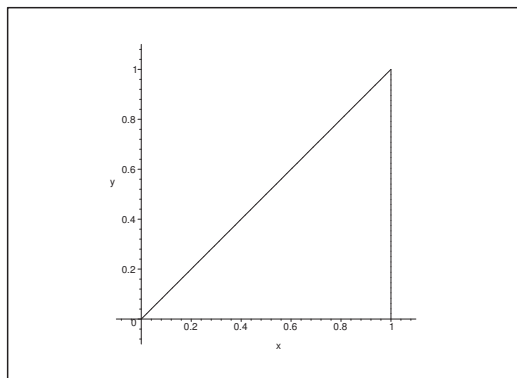


Figure 68: The parametric domain  $E$ .

I. 1) The parametric domain is

$$E = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq u\}.$$

2) Next we shall find the normal vector of the surface in the given parametric representation. (Note that we still do not consider the vector field itself). Since the tangent vectors of the coordinate curves are

$$\frac{\partial \mathbf{r}}{\partial u} = (\sin v, \cos v, 0) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = u(\cos v, -\sin v, 1),$$

the corresponding normal vector is

$$\begin{aligned} \mathbf{N}(u, v) &= \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = u \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \sin v & \cos v & 0 \\ \cos v & -\sin v & 1 \end{vmatrix} \\ &= u(\cos v + v \sin v, v \cos v - \sin v, -1). \end{aligned}$$

- 3) The vector field  $\mathbf{V}(x, y, z)$  is defined in the whole of  $\mathbb{R}^3$ . Its restriction to the surface  $\mathcal{F}$  is expressed in the new variables  $(u, v)$

$$\begin{aligned}\mathbf{V}(x, y, z) &= (yz, -xz, x^2 + y^2) \\ &= (u \cos v \cdot uv, -u \sin v \cdot uv, (u \sin^2 v)^2 + (u \cos^2 v)^2) \\ &= (u^2 v \cos v, -u^2 v \sin v, u^2) = u^2 (v \cos v, -v \sin v, 1).\end{aligned}$$

After these opening manoeuvres continue with

- 4) Calculation of the *integrand* expressed by  $(u, v)$  (apply 2 and 3 above):

$$\begin{aligned}\mathbf{V}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v) &= u^2 (v \cos v, -v \sin v, 1) \cdot u(\cos v + v \sin v, v \cos v - \sin v, -1) \\ &= u^3 \{v \cos v(\cos v + v \sin v) - v \sin v(v \cos v - \sin v) - 1\} \\ &= u^3 \{v(\cos^2 v + v \cos v \sin v - v \cos v \sin v + \sin^2 v) - 1\} \\ &= u^3(v - 1).\end{aligned}$$

- 5) Calculation of the *flux* (apply 1 and 4):

$$\begin{aligned}\Phi_1 &= \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS = \int_E \mathbf{V}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v) \, du \, dv \\ &= \int_E u^3(v - 1) \, du \, dv = \int_0^1 u^3 \left\{ \int_0^u (v - 1) \, dv \right\} du \\ &= \int_0^1 u^3 \left[ \frac{1}{2} v^2 - v \right]_0^u du = \int_0^1 u^3 \left( \frac{1}{2} u^2 - u \right) du \\ &= \int_0^1 \left( \frac{1}{2} u^5 - u^4 \right) du = \left[ \frac{1}{2} \cdot \frac{1}{6} u^6 - \frac{1}{5} u^5 \right]_0^1 \\ &= \frac{1}{12} - \frac{1}{5} = -\frac{7}{60}. \quad \diamond\end{aligned}$$

### Example 9.4

**A.** Find the flux  $\Phi_2$  of the vector field

$$\mathbf{V}(x, y, z) = (x^2 + y^2, z^2, y^2), \quad (x, y, z) \in \mathbb{R}^3,$$

through the surface  $\mathcal{F}$  defined by

$$\mathbf{r}(u, v) = (u + v, u - v, u + 2v), \quad u^2 + v^2 \leq 4.$$

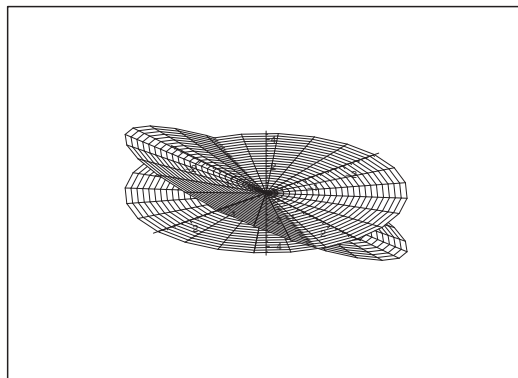


Figure 69: The surface  $\mathcal{F}$  and its projection onto the  $(x, y)$ -plane.



- D. We see that the surface  $\mathcal{F}$  lies in a plane, but because this plane is oblique, it is very difficult to exploit its flat structure. Instead we analyze the reduction formula

$$\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS = \int_E \mathbf{V}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v) du dv,$$

where the abstract *surface* integral is rewritten as an abstract *plane* integral. By inspecting the right hand side it is seen that we shall

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- 1) identify the parametric domain  $E$ ,
- 2) find the normal vector  $\mathbf{N}(u, v)$  for the surface  $\mathcal{F}$ , corresponding to the parameters  $(u, v)$ ,
- 3) express  $\mathbf{V}(\mathbf{r}(u, v))$  on the surface  $\mathcal{F}$  as a function in the parameters  $(u, v)$ .

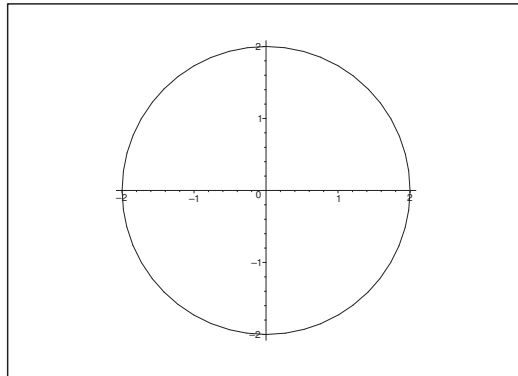


Figure 70: The parametric domain  $E$  is a disc of centre  $(0, 0)$  and radius 2.

- I. 1) *The parametric domain* is the disc of centre  $(0, 0)$  and radius 2,

$$E = \{(u, v) \mid u^2 + v^2 \leq 4 = 2^2\}.$$

- 2) *The normal vector.* It follows from the parametric representation of the surface that

$$\frac{\partial \mathbf{r}}{\partial u} = (1, 1, 1) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = (1, -1, 2).$$

Thus the normal vector is

$$\mathbf{N}(u, v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = (3, -1, -2).$$

- 3) *The restriction of the vector field* to the surface is now given by

$$\begin{aligned} \mathbf{V}(x, y, z) &= (x^2, y^2, z^2) \\ &= ((u+v)^2 + (u-v)^2, (y+2v)^2, (u-v)^2) \\ &= (2(u^2 + v^2), (u+2v)^2, (u-v)^2). \end{aligned}$$

- 4) *The integrand* is according to 2 and 3,

$$\begin{aligned} \mathbf{V} \cdot \mathbf{N} &= (3, -1, -2) \cdot (2(u^2 + v^2), (u+2v)^2, (u-v)^2) \\ &= 6(u^2 + v^2) - (u+2v)^2 - 2(u-v)^2 \\ &= 6u^2 + 6v^2 - (u^2 + 4uv + 4v^2) - (2u^2 - 4uv + 2v^2) \\ &= 3u^2. \end{aligned}$$

5) By *insertion* of 4 in the reduction formula we get by also using 1,

$$\begin{aligned} \Phi_2 &= \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS = \int_E \mathbf{V}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v) du dv \\ &= 3 \int_E u^2 du dv. \end{aligned}$$

Since the parametric domain  $E$  is a *disc*, it is easiest to reduce it in *polar* coordinates,

$$u = \rho \cos \varphi, \quad v = \rho \sin \varphi, \quad 0 \leq \rho \leq 2, \quad 0 \leq \varphi \leq 2\pi.$$

Hence we get the result

$$\begin{aligned} \Phi_2 &= 3 \int_E u^2 du dv = 3 \int_0^{2\pi} \left\{ \int_0^2 \rho^2 \cos^2 \varphi \cdot \rho d\rho \right\} d\varphi \\ &= 3 \int_0^{2\pi} \cos^2 \varphi d\varphi \cdot \int_0^2 \rho^3 d\rho = 3 \cdot \pi \cdot \left[ \frac{1}{4} \rho^4 \right]_0^2 \\ &= 12\pi. \quad \diamond \end{aligned}$$

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**Example 9.5**

A. Let  $a, b, c > 0$ , be constants, and let

$$\mathbf{V}(x, y, z) = (y, x, z + c), \quad (x, y, z) \in \mathbb{R}^3.$$

Find the flux  $\Phi_3$  of  $\mathbf{V}$  through the half ellipsoidal surface

$$\mathcal{F}_1 = \left\{ (x, y, z) \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1, z \geq 0 \right\}$$

where the normal is directed upwards,  $\mathbf{n} \cdot \mathbf{e}_z \geq 0$ , and the flux  $\Phi_4$  of  $\mathbf{V}$  through the projection  $\mathcal{F}_2$  of  $\mathcal{F}_1$  onto the  $(x, y)$ -plane,

$$\mathcal{F}_2 = \left\{ (x, y, z) \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1 \right\}, \quad \mathbf{n} = (0, 0, 1).$$

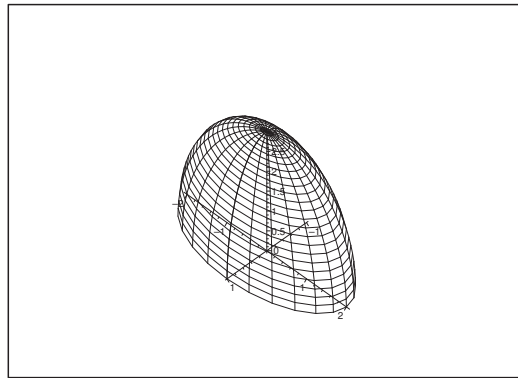


Figure 71: The half ellipsoidal surface  $\mathcal{F}_1$  for  $a = 1$ ,  $b = 2$  and  $c = 3$ . The surface  $\mathcal{F}_2$  is hidden below  $\mathcal{F}_1$  in the  $(x, y)$ -plane.

D. Summing up we see that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  surround a spatial domain  $\Omega$ . The flux  $\Phi_3$  represents e.g. the energy which flows out of  $\Omega$  through  $\mathcal{F}_1$ , and  $\Phi_4$  represents the energy which flows into  $\Omega$  through  $\mathcal{F}_2$ . Hence, the difference  $\Phi_3 - \Phi_4$  represents the energy which is created by  $\mathbf{V}$  in  $\Omega$ .

I 1. Consider first

$$\mathcal{F}_1 = \left\{ (x, y, z) \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1, z \geq 0 \right\}, \quad \mathbf{n} \cdot \mathbf{e}_3 \geq 0.$$

The easiest method, which can be found in some textbooks, is to use *spherical* coordinates (left to the reader). We shall here as an *alternative* apply *rectangular* coordinates instead. Then we can consider  $\mathcal{F}_1$  as the *graph* of the function

$$z = f(x, y) = c \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}, \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1.$$

Then the hidden parametric representation is given by

$$\mathbf{r}(x, y) = \left( x, y, c \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} \right), \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1.$$

This parametric representation is differentiable when

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1,$$

i.e. when  $z > 0$ . If so, we get

$$\frac{\partial \mathbf{r}}{\partial x} = \left( 1, 0, -\frac{c}{a^2} \frac{x}{\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}} \right) = \left( 1, 0, -\frac{c^2}{a^2} \cdot \frac{x}{z} \right),$$

and analogously

$$\frac{\partial \mathbf{r}}{\partial y} = \left( 0, 1, -\frac{c}{b^2} \frac{y}{\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}} \right) = \left( 0, 1, -\frac{c^2}{b^2} \cdot \frac{y}{z} \right),$$

where we have used that  $z = c \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}$  in order not to be overburdened with a square root in the following. (It is always possible to substitute back again, if necessary). Then

$$\mathbf{N}(x, y) = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & -\frac{c^2}{a^2} \frac{x}{z} \\ 0 & 1 & -\frac{c^2}{b^2} \frac{y}{z} \end{vmatrix} = \left( \frac{c^2}{a^2} \cdot \frac{x}{z}, \frac{c^2}{b^2} \cdot \frac{y}{z}, 1 \right).$$

Now  $\mathbf{N} \cdot \mathbf{e}_3 = 1 > 0$ , so  $\mathbf{N}(x, y)$  is pointing in the right direction.

The integrand is then calculated,

$$\begin{aligned} \mathbf{V} \cdot \mathbf{N} &= (y, x, z + c) \cdot \left( \frac{c^2}{a^2} \cdot \frac{x}{z}, \frac{c^2}{b^2} \cdot \frac{y}{z}, 1 \right) \\ &= c^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \frac{xy}{z} + z + c. \end{aligned}$$

The domain of integration is the ellipse in the  $(x, y)$ -plane

$$E = \left\{ (x, y) \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1 \right\}.$$

Hence, the flux is equal to the *improper* plane integral

$$\begin{aligned}\Phi_3 &= \int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} dS \\ &= \int_E \left\{ c \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \cdot \frac{xy}{\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}} + c \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} + c \right\} dx dy.\end{aligned}$$

Then notice that we have e.g.

$$\int_x \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} dx = -a^2 \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2},$$

i.e. if we integrate over an interval of the form  $[0, k]$  (where the integrand is  $\geq 0$ ) or over  $[-k, 0]$  (where the integrand is  $\leq 0$ ), then we get *finite* values in both cases, i.e. the improper integral is convergent.

If we put  $k = a\sqrt{1 - \left(\frac{y}{b}\right)^2}$ , it follows of symmetric reasons that

$$\int_E c \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \frac{xy}{\sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}} dS$$

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$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0^+} \int_{-b}^b c \left( \frac{1}{a^2} + \frac{1}{b^2} \right) y \left\{ \int_{-a\sqrt{1-(\frac{y}{b})^2+\varepsilon}}^{a\sqrt{1-(\frac{y}{b})^2-\varepsilon}} \frac{x dx}{\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}} \right\} \\
&= \lim_{\varepsilon \rightarrow 0^+} c \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \int_{-b}^b y \cdot 0 dy = 0.
\end{aligned}$$

The expression of the flux is therefore reduced to

$$\begin{aligned}
\Phi_3 &= 0 + \int_E (z + c) dx dy \\
&= \int_E c \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} dx dy + c \cdot \text{area}(E) \\
&= c \int_E \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} dx dy + c \cdot \pi ab.
\end{aligned}$$

The purpose of the following *elaborated* variant is to straighten up the ellipse by the change of variables

$$u = \frac{x}{a}, \quad v = \frac{y}{b}, \quad \text{dvs.} \quad x = au, \quad y = bv.$$

The corresponding Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab > 0.$$

By the transformation formula the parametric domain  $E$  is mapped into the unit disc  $B$  in the  $(u, v)$ -plane, hence

$$\begin{aligned}
\Phi_3 &= \pi abc + c \int_E \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} dx dy \\
&= \pi abc + c \int_B \sqrt{1 - u^2 - v^2} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \\
&= \pi abc + ab \cdot c \int_B \sqrt{1 - u^2 - v^2} du dv \\
&= abc \left\{ \pi + \int_0^{2\pi} \left\{ \int_0^1 \sqrt{1 - \varrho^2} \cdot \varrho d\varrho \right\} d\varphi \right\} \\
&= abc \left\{ \pi + 2\pi \int_0^1 \sqrt{1 - t} \cdot \frac{1}{2} dt \right\} \\
&= \pi abc \left\{ 1 + \left[ -\frac{2}{3} (1 - t)^{\frac{3}{2}} \right]_0^1 \right\} \\
&= \pi abc \cdot \left( 1 + \frac{2}{3} \right) = \frac{5}{3} \pi abc.
\end{aligned}$$

No matter whether one is using spherical or rectangular coordinates, it is very difficult to find  $\Phi_3$ , and there are lots of pit holes (as seen above we get e.g. an improper surface integral in the rectangular version).

**I 2.** Next look at

$$\mathcal{F}_2 = E = \left\{ (x, y, z) \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1, z = 0 \right\}, \quad \mathbf{n} = (0, 0, 1).$$

The restriction of  $\mathbf{V}$  to  $E$  is obtained by putting  $z = 0$ , i.e.

$$\mathbf{V}(x, y, 0) = (y, x, c).$$

The unit normal vector is  $\mathbf{n} = (0, 0, 1)$ , so the integrand becomes

$$\mathbf{V}(x, y, 0) \cdot \mathbf{n} = (y, x, c) \cdot (0, 0, 1) = c.$$

We conclude by using the reduction theorem on the simple calculation

$$\Phi_4 = \int_E \mathbf{V} \cdot \mathbf{n} dS = c \int_E dS = c \cdot \text{areal}(E) = c \cdot \pi ab = \pi abc.$$

**I 3.** Finally we have (cf. the figure)

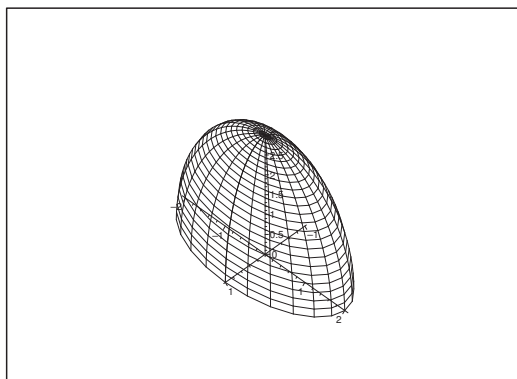


Figure 72: The domain  $\Omega$ .

The flux *out of*  $\partial\Omega$  of  $\mathbf{V}$  is according to **I 1.** and **I 2.** given by

$$\Phi_3 - \Phi_4 = \frac{5}{3} \pi abc - \pi abc = \frac{2}{3} \pi abc,$$

where we use  $-\Phi_4$ , because  $\Phi_4$  indicates the flux *into*  $\Omega$  through  $\mathcal{F}_2$ .

Let us now **alternatively** show the same result by means of *Gauß's theorem*.

We first realize that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  surround a spatial domain

$$\Omega = \left\{ (x, y, z) \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1, z \geq 0 \right\}.$$

From  $\mathbf{V}(x, y, z) = (y, x, z + c)$  we then get

$$\text{div } \mathbf{V} = 0 + 0 + 1 = 1.$$



The flux out through  $\partial\Omega$  (the normal of direction away from the domain) is then according to *Gauß's theorem*,

$$\Phi = \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} dS = \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = \int_{\Omega} d\Omega = \operatorname{vol}(\Omega) = \frac{1}{2} \left( \frac{4\pi}{3} abc \right) = \frac{2\pi}{3} abc.$$

By comparison we see that this is exactly  $\Phi_3 - \Phi_3$  as we claimed.

Summarizing,  $\Phi_3$  in **I 1.** was difficult to compute, while  $\Phi_4$  in **I 2.** and  $\Phi$  in **I 3.** were easy. Since  $\Phi_3 - \Phi_4 = \Phi$ , we might have calculated  $\Phi_3$  by computing the easy right hand side of

$$\Phi_3 = \Phi_4 + \Phi,$$

i.e. expressed in integrals,

$$(14) \int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} dS = \int_{\mathcal{F}_2} \mathbf{V} \cdot \mathbf{n} dS + \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega,$$

or put in other words: an ugly surface integral (the left hand side) is expressed by the sum of a simple surface integral (here even a plane integral) and a simple spatial integral (the right hand side).

This technique can often be applied when one shall calculate the flux through a more or less complicated surface  $\mathcal{F}_1$ .

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- 1) First draw a figure, thereby realizing how  $\mathcal{F}_1$  is placed in the space.
- 2) Then add a nice surface  $\mathcal{F}_2$ , such that  $\mathcal{F}_1 \cup \mathcal{F}_2$  becomes the boundary of a spatial body  $\Omega$ . Check in particular that the normal vector on  $\mathcal{F}_2$  is always pointing *away* from the domain  $\Omega$ ).
- 3) Calculate the right hand side of (14), thereby finding the flux through  $\mathcal{F}_1$ .

**Remark 9.2** We shall later in Example 9.10 give some comments which will give us an even more easy version of calculation.  $\diamond$

**Example 9.6**

**A.** Let the surface  $\mathcal{F}$  be the square

$$\mathcal{F} = \{(x, y, z) \mid |x| \leq a, |y| \leq a, z = a\}, \quad a > 0,$$

at the height  $a$  with the unit normal vector  $\mathbf{n} = (0, 0, 1)$  pointing upwards.

Find the flux through  $\mathcal{F}$  of the Coulomb field

$$\mathbf{V}(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}, \quad (x, y, z) \neq (0, 0, 0).$$

(Concerning the Coulomb field see also Example 9.10).

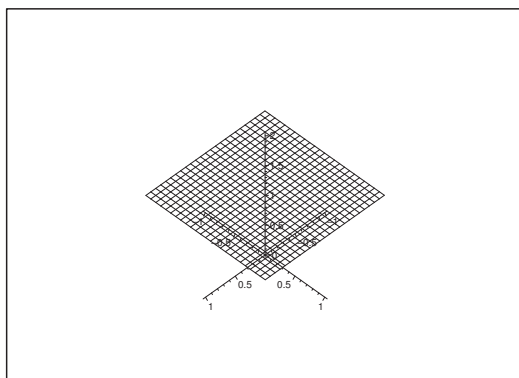


Figure 73: The surface  $\mathcal{F}$  for  $a = 1$ .

**D.** Using *rectangular* coordinates we get from the reduction theorem that

$$\begin{aligned} \Phi_5 &= \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS = \int_{-a}^a \left\{ \int_{-a}^a \frac{(x, y, a) \cdot (0, 0, 1)}{(x^2 + y^2 + a^2)^{3/2}} \, dx \right\} dy \\ &= \int_{-a}^a \left\{ \frac{a}{(x^2 + y^2 + a^2)^{3/2}} \, dx \right\} dy = 4a \int_0^a \left\{ \int_0^a \frac{1}{(x^2 + y^2 + a^2)^{3/2}} \, dx \right\} dy, \end{aligned}$$

where we have used that the integrand is even in both  $x$  and  $y$ , and that the domain is symmetric.

So far, so good, but from now on the calculations become really tough. The reason is that the integrand invites to the application of *polar* coordinates, while the domain is better described in *rectangular* coordinates. The mixture of these two coordinate systems will always cause some difficulties.

For pedagogical reasons we shall here show both variants, first the *rectangular version*, which is extremely difficult, and afterwards the *polar version*, which is “only” difficult. This exercise will show that one cannot just restrict oneself to only learning the rectangular method!

I 1. *Rectangular variant.* Calculate directly

$$\Phi_5 = 4 \int_0^a \left\{ \int_0^a \frac{a}{(x^2 + y^2 + a^2)^{3/2}} dx \right\} dy.$$

First we note by a partial integration of auxiliary function that

$$\begin{aligned} \int_0^a 1 \cdot (t^2 + c^2)^\alpha dt &= \left[ t \cdot (t^2 + c^2)^\alpha \right]_0^a - \int_0^a t \cdot \alpha (t^2 + c^2)^{\alpha-1} \cdot 2t dt \\ &= a (a^2 + c^2)^\alpha - 2\alpha \int_0^a (t^2 + c^2 - c^2) (t^2 + c^2)^{\alpha-1} dt \\ &= a (a^2 + c^2)^\alpha - 2\alpha \int_0^a (t^2 + c^2)^\alpha dt + 2\alpha c^2 \int_0^a (t^2 + c^2)^{\alpha-1} dt. \end{aligned}$$

When  $\alpha \neq 0$  and  $c > 0$ , we get by a rearrangement

$$(15) \int_0^a (t^2 + c^2)^{\alpha-1} dt = \frac{1 + 2\alpha}{2\alpha c^2} \int_0^a (t^2 + c^2)^\alpha dt - \frac{a (a^2 + c^2)^\alpha}{2\alpha c^2}.$$

Choosing  $t = x$  and  $\alpha = -\frac{1}{2}$  and  $c^2 = y^2 + a^2$  in (15) and multiplying by  $a$ , we get the inner integral in  $\Phi_5$ : Since  $1 + 2\alpha = 0$  we have

$$\int_0^a \frac{a}{(x^2 + y^2 + a^2)^{3/2}} dx = -\frac{a^2 (a^2 + y^2 + a^2)^{-1/2}}{2 \left(-\frac{1}{2}\right) (a^2 + y^2)} = \frac{a^2}{(y^2 + a^2) \sqrt{y^2 + 2a^2}},$$

which gives by insertion

$$\Phi_5 = 4 \int_0^a \frac{a^2}{(y^2 + a^2) \sqrt{y^2 + 2a^2}} dy.$$

So far we can still use the pocket calculator TI-89, but from now on it denies to calculate the exact value! Therefore, we must from now on continue by using the old-fashioned, though well tested methods from the time before the pocket calculators.

When we consider the dimensions we see that  $y \sim a$ , hence a convenient substitution must be  $y = au$ . Then

$$\begin{aligned} \Phi_5 &= 4 \int_0^a \frac{a^2}{(y^2 + a^2) \sqrt{y^2 + 2a^2}} dy = 4 \int_0^1 \frac{a^2}{(a^2 u^2 + a^2) \sqrt{a^2 u^2 + 2a^2}} \cdot a du \\ &= 4 \int_0^1 \frac{1}{(u^2 + 1) \sqrt{u^2 + 2}} du = 4 \int_0^1 \frac{1}{(u^2 + 1) \sqrt{(u^2 + 1) + 1}} du, \end{aligned}$$

where it should be surprising that  $\Phi_5$  is *independent of*  $a$ .

The following circumscription is governed by the following general principle:

- Whenever the square of two terms is involved then it should be rewritten as 1 plus/minus something which has “something to do” with the other terms in the integrand.

The circumscription indicates that we should try the *monotonous* substitution

$$t = u^2 + 1, \quad u = \sqrt{t-1}, \quad du = \frac{1}{2} \frac{1}{\sqrt{t-1}} dt, \quad t \in [1, 2].$$

By this substitution we get

$$\Phi_5 = 4 \int_0^1 \frac{1}{(u^2 + 1) \sqrt{(u^2 + 1) + 1}} du = 4 \int_1^2 \frac{1}{t\sqrt{t+1}} \cdot \frac{1}{2} \frac{1}{\sqrt{t-1}} dt = 2 \int_1^2 \frac{dt}{t\sqrt{t^2-1}}.$$

The structure  $\sqrt{t^2-1}$  looks like

$$\sqrt{\cosh^2 w - 1} = \sqrt{\sinh^2 w} = |\sinh w|,$$

which is a means to get rid of the square root. We therefore try another substitution,

$$t = \cosh w, \quad w = \ln(t + \sqrt{t^2-1}), \quad dt = \sinh w dw, \quad w \in [0, \ln(2 + \sqrt{3})].$$

Since we have  $\sinh w \geq 0$  in this interval, we get

$$\begin{aligned} \Phi_5 &= 2 \int_1^2 \frac{dt}{t\sqrt{t^2-1}} = 2 \int_0^{\ln(2+\sqrt{3})} \frac{\sinh w}{\cosh w \cdot \sinh w} dw \\ &= 2 \int_0^{\ln(2+\sqrt{3})} \frac{dw}{\cosh w} = 2 \int_0^{\ln(2+\sqrt{3})} \frac{2}{e^w + e^{-w}} dw \\ &= 4 \int_0^{\ln(2+\sqrt{3})} \frac{e^w}{1 + (e^w)^2} dw = 4 [\text{Arctan}(e^w)]_0^{\ln(2+\sqrt{3})} \\ &= 4 \left\{ \text{Arctan}(2 + \sqrt{3}) - \frac{\pi}{4} \right\} = 4 \text{Arctan}(2 + \sqrt{3}) - \pi. \end{aligned}$$

Our troubles in the rectangular case are not over. How can we find  $\text{Arctan}(2 + \sqrt{3})$  without using a pocket calculator?

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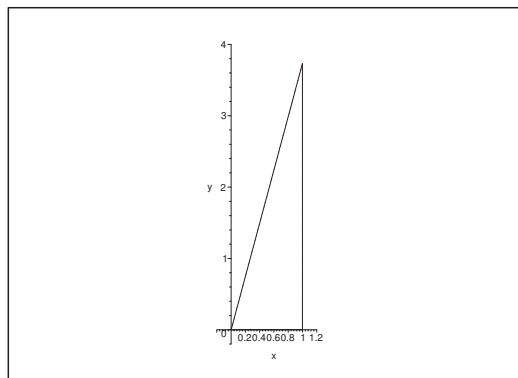


Figure 74: The rectangular triangle with the opposite side =  $2 + \sqrt{3}$ , so  $\varphi = \text{Arctan}(2 + \sqrt{3})$  is the nearby angle.

Geometrically  $\varphi = \text{Arctan}(2 + \sqrt{3})$  is that angle in the rectangular triangle on the figure, which is  $> \frac{\pi}{4}$ .

The hypotenuse can be found by Pythagoras' theorem,

$$r^2 = (2 + \sqrt{3})^2 + 1^2 = 4 + 3 + 4\sqrt{3} + 1 = 8 + 4\sqrt{3} = 4(2 + \sqrt{3}),$$

i.e.  $r = 2\sqrt{2 + \sqrt{3}}$ . From  $\varphi > \frac{\pi}{4}$ , follows that  $\psi = \frac{\pi}{2} - \varphi < \frac{\pi}{4}$ , and

$$\cos \psi = \frac{1}{r} (2 + \sqrt{3}) = \frac{1}{2} \sqrt{2 + \sqrt{3}}.$$

We shall get rid of the square root by squaring, so we try

$$\cos 2\psi = 2 \cos^2 \psi - 1 = 2 \cdot \frac{1}{4} (2 + \sqrt{3}) - 1 = 1 + \frac{1}{2} \sqrt{3} - 1 = \frac{\sqrt{3}}{2}.$$

Since we have been so careful to show that  $0 < \psi < \frac{\pi}{4}$ , it follows that  $0 < 2\psi < \frac{\pi}{2}$ , hence

$$2\psi = \text{Arccos} \left( \frac{\sqrt{3}}{2} \right) = \frac{\pi}{6}, \quad \text{i.e. } \psi = \frac{\pi}{12}.$$

Then

$$\text{Arctan}(2 + \sqrt{3}) = \varphi = \frac{\pi}{2} - \psi = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}.$$

By a final insertion we get that the flux is

$$\Phi_5 = 4 \text{Arctan}(2 + \sqrt{3}) - \pi = 4 \cdot \frac{5\pi}{12} - \pi = \frac{5\pi}{3} - \pi = \frac{2\pi}{3}.$$

**Remark 9.3** It is obvious why this variant is never seen in ordinary textbooks. The morale is that even if something can be done, it does not always have to, and we should of course have avoided this variant. It should, however, be added that the pocket calculator finally will find that

$$\text{Arctan}(2 + \sqrt{3}) = \frac{5\pi}{12}. \quad \diamond$$

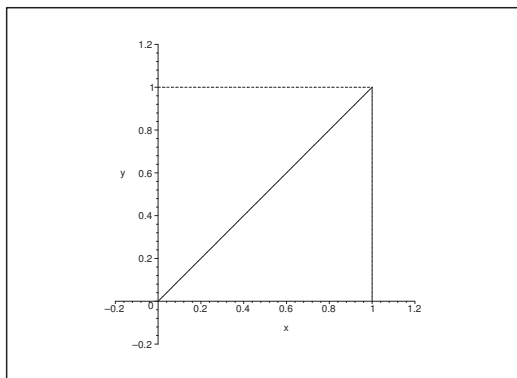


Figure 75: The domain  $T$  is the lower triangle and  $a = 1$ .

**I 2. Polar variant.** We shall start from the very beginning by

$$\Phi_5 = 4 \int_0^a \left\{ \int_0^a \frac{a}{(x^2 + y^2 + a^2)^{3/2}} dx \right\} dy.$$

The domain  $[0, a]^2$  is not fit for a polar description, but if we note that the integrand is symmetrical about the line  $y = x$ , then this symmetry gives that

$$\Phi_5 = 2 \cdot 4 \int_T \frac{a}{(x^2 + y^2 + a^2)^{3/2}} dx dy = 8 \int_T \frac{a}{(x^2 + y^2 + a^2)^{3/2}} dx dy,$$

where the triangle  $T$  is bounded by  $y = 0$  in the right half-plane (corresponding in polar coordinates to  $\varphi = 0$ ), the line  $y = x$  (corresponding to  $\varphi = \frac{\pi}{4}$ ) and  $x = \varrho \cos \varphi = a$ , i.e.

$$\varrho = \frac{a}{\cos \varphi}.$$

Therefore, a *polar* description of  $T$  is

$$T = \left\{ (\varrho, \varphi) \mid 0 \leq \varphi \leq \frac{\pi}{4}, 0 \leq \varrho \leq \frac{a}{\cos \varphi} \right\}.$$

Then the rest follows from the usual reduction theorems,

$$\begin{aligned} \Phi_5 &= 8 \int_T \frac{a}{(x^2 + y^2 + a^2)^{3/2}} dS \\ &= 8 \int_0^{\frac{\pi}{4}} \left\{ \int_0^{\frac{a}{\cos \varphi}} \frac{a}{(\varrho^2 + a^2)^{3/2}} \cdot \varrho d\varrho \right\} d\varphi. \end{aligned}$$

The inner integral is calculated by using the substitution

$$t = \varrho, \quad dt = 2\varrho d\varrho, \quad \text{i.e. } \varrho d\varrho = \frac{1}{2} dt,$$

hence

$$\begin{aligned}
 8 \int_0^{\frac{a}{\cos \varphi}} \frac{a}{(\varrho^2 + a^2)^{3/2}} \varrho d\varrho &= 4 \int_0^{\frac{4}{\cos^2 \varphi}} \frac{a}{(t + a^2)^{3/2}} dt \\
 &= 4 \left[ -\frac{2a}{(t + a^2)^{1/2}} \right]_{\frac{a^2}{\cos^2 \varphi}}^{\frac{4}{\cos^2 \varphi}} = 8 \left\{ 1 - \frac{a}{\sqrt{\frac{a^2}{\cos^2 \varphi} + a^2}} \right\} \\
 &= 8 \left\{ 1 - \frac{a |\cos \varphi|}{a \sqrt{1 + \cos^2 \varphi}} \right\} = 8 \left\{ 1 - \frac{|\cos \varphi|}{\sqrt{1 + \cos^2 \varphi}} \right\}.
 \end{aligned}$$


Since  $|\cos \varphi| = \cos \varphi$  for  $0 \leq \varphi \leq \frac{\pi}{4}$ , we get by an insertion and an application of the substitution

$$u = \sin \varphi, \quad du = \cos \varphi d\varphi, \quad \cos^2 \varphi = 1 - \sin^2 \varphi = 1 - u^2,$$

that


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$$\begin{aligned}
\Phi_5 &= 8 \int_0^{\frac{\pi}{4}} \left\{ 1 - \frac{1}{\sqrt{1 + \cos^2 \varphi}} \cdot \cos \varphi \right\} d\varphi \\
&= 8 \cdot \frac{\pi}{4} - 8 \int_0^{\frac{1}{\sqrt{2}}} \frac{du}{\sqrt{1 + (1 - u^2)}} = 2\pi - 8 \int_0^{\frac{1}{\sqrt{2}}} \frac{du}{\sqrt{2 - u^2}} \\
&= 2\pi - 8 \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1 - \left(\frac{u}{\sqrt{2}}\right)^2}} \cdot \frac{1}{\sqrt{2}} du = 2\pi - 8 \int_0^{\frac{1}{2}} \frac{dv}{\sqrt{1 - v^2}} \\
&= 2\pi - 8 [\text{Arcsin } v]_0^{\frac{1}{2}} = 2\pi - 8 \cdot \frac{\pi}{6} = \frac{2\pi}{3}.
\end{aligned}$$

**Remark 9.4** It should be admitted that the polar version also contains some difficulties, though they are not as bad as in the rectangular version.  $\diamond$

### Example 9.7

Let  $\mathbf{V}(x, y, z) = (x, y, z)$  be the *vector field of location*, and let  $\Omega$  be any spatial domain. Then

$$\text{div } \mathbf{V} = 1 + 1 + 1 = 3,$$

is the flux of  $\mathbf{V}$  through  $\partial\Omega$  according to *Gauß's theorem*

$$\Phi = \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} dS = \int_{\Omega} \text{div } \mathbf{V} d\Omega = 3 \text{ vol}(\Omega).$$

As an unexpected result we see that we can express  $\text{vol}(\Omega)$  as a *surface integral*

$$\text{vol}(\Omega) = \frac{1}{3} \int_{\partial\Omega} \mathbf{x} \cdot \mathbf{n} dS.$$

Since  $\text{div}(x_1, \dots, x_n) = n$  only depends on the dimension, we get a similar *area formula* expressed by a *line integral*

$$\text{areal}(E) = \frac{1}{2} \int_{\partial E} \mathbf{x} \cdot \mathbf{n} ds. \quad \diamond$$

**Example 9.8** Let  $\mathbf{V}(x, y, z) = (\alpha_1, \alpha_2, \alpha_3) = \alpha$  be a constant vector field, and let  $\Omega$  be any spatial domain. Then the flux through  $\partial\Omega$  is given by

$$\int_{\partial\Omega} \alpha \cdot \mathbf{n} dS = 0,$$

which either can be proved by *Gauß's theorem*,

$$\int_{\partial\Omega} \alpha \cdot \mathbf{n} dS = \int_{\Omega} \text{div } \alpha d\Omega = \int_{\Omega} 0 d\Omega = 0,$$

or by common sense, because if the vector field is constant then there must flow just as much into the domain as is flowing out again in time.

Note, however, the difference in the proofs. The latter argument is intuitive, while *Gauß's theorem* gives the correct proof. However, it is always nice to get one's intuition confirmed.  $\diamond$



**Example 9.9**

The following example gives an important result.

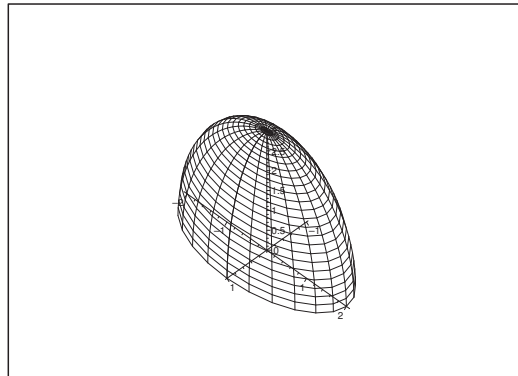


Figure 76: An example of a domain  $\Omega$  where there is an obvious splitting of the surface  $\partial\Omega$  into two sub-surfaces  $\mathcal{F}_1$  and  $\mathcal{F}_2$ .

Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be two surfaces which together surrounds a spatial domain  $\Omega$ . Let  $\mathbf{n}$  be the outer vector field of unit normal vectors. Finally, let  $\mathbf{V}(x, y, z)$  be a vector field of class  $C^1$  in an open set which contains  $\Omega$ .

According to *Gauß's theorem* the flux through  $\partial\Omega$  is given by

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} dS = \int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} dS + \int_{\mathcal{F}_2} \mathbf{V} \cdot \mathbf{n} dS = \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega.$$

We get by a rearrangement

$$\begin{aligned} \int_{\mathcal{F}_2} \mathbf{V} \cdot \mathbf{n} dS &= \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega - \int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} dS \\ (16) \qquad \qquad &= \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega + \int_{\mathcal{F}_1} \mathbf{V} \cdot (-\mathbf{n}) dS. \end{aligned}$$

Hence, what flows out of  $\Omega$  by the vector field  $\mathbf{V}$  through the surface  $\mathcal{F}_2$  is equal to what flows into  $\Omega$  by  $\mathbf{V}$  through  $\mathcal{F}_1$  plus what  $\mathbf{V}$  creates in the domain  $\Omega$ .

Since what is created by  $\mathbf{V}$  in  $\Omega$ , i.e.  $\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega$ , often is easy to calculate, we have got a method for calculation of the flux through a complicated surface  $\mathcal{F}_2$  by using formula (16).

In the case where  $\mathbf{V}$  is divergence free,  $\operatorname{div} \mathbf{V} = 0$ , formula (16) becomes particularly simple,

$$\int_{\mathcal{F}_2} \mathbf{V} \cdot \mathbf{n} dS = \int_{\mathcal{F}_1} \mathbf{V} \cdot (-\mathbf{n}) dS.$$

Since  $\mathbf{n}$  on  $\mathcal{F}_2$  and  $-\mathbf{n}$  on  $\mathcal{F}_1$  point in the “same direction”, this shows that the surface of integration  $\mathcal{F}_2$  can be deformed freely, when  $\operatorname{div} \mathbf{V} = 0$ , if only the boundary curve  $\delta\mathcal{F}_2$  with its direction of revolution is kept fixed.  $\diamond$

**Example 9.10**

The Coulomb vector field (cf. Example 9.6),

$$\mathbf{V}(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}, \quad \text{for } (x, y, z) \neq (0, 0, 0),$$

satisfies (where one absolutely should *not* put everything in the same fraction with the same denominator, unless one wants to obscure everything)

$$\begin{aligned} \frac{\partial V_x}{\partial x} &= \frac{\partial}{\partial x} \left\{ \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right\} = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}}, \\ \frac{\partial V_y}{\partial y} &= \frac{\partial}{\partial y} \left\{ \frac{y}{(x^2 + y^2 + z^2)^{3/2}} \right\} = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3y^2}{(x^2 + y^2 + z^2)^{5/2}}, \\ \frac{\partial V_z}{\partial z} &= \frac{\partial}{\partial z} \left\{ \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right\} = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3z^2}{(x^2 + y^2 + z^2)^{5/2}}, \end{aligned}$$

from which we get by adding these expressions,

$$\operatorname{div} \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \frac{3}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} = 0,$$

i.e.  $\mathbf{V}$  is divergence free so we can use Example 9.9 for domains  $\Omega$ , which do *not* contain the point  $(0, 0, 0)$ .

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- A. Let  $\Omega$  be any spatial domain with  $(0,0,0)$  as an inner point. Find the flux of the Coulomb field through  $\partial\Omega$ , i.e. find  $\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} dS$ .
- D. Since  $\operatorname{div} \mathbf{V}$  is *not* defined in  $(0,0,0)$ , we cannot apply Gauß's theorem directly. But since  $(0,0,0)$  is an inner point, there exists a ball

$$K = K(\mathbf{0}; r) \subset \Omega,$$

totally contained in  $\Omega$ . If we cut  $K$  out of  $\Omega$ , we get a domain  $\tilde{\Omega} = \Omega \setminus K$ , in which  $\operatorname{div} \mathbf{V}$  is defined everywhere and equal to 0. According to Example 9.9 the surface  $\partial\Omega$  can be deformed into  $\partial K$ , and then the flux through  $\partial K$  can be calculated as an ordinary surface integral. (The singular point  $(0,0,0)$  lies in  $K$ , so we cannot apply Gauß's theorem in the latter calculation).

- I. We are just missing one thing. Since both  $\partial\Omega$  and  $\partial K$  are closed surfaces, neither of them has a boundary curve, so we get formally

$$\delta(\partial\Omega) = \emptyset = \delta(\partial K).$$

*Alternatively* there is flowing just as much into  $\tilde{\Omega}$  through  $\partial K$  as out of  $\tilde{\Omega}$  through  $\partial\Omega$ , because the flow is balanced.

Thus we have proved by using Gauß's theorem that

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} dS = \int_{\partial K} \mathbf{V} \cdot \mathbf{n} dS.$$

The right hand side is calculated as a usual surface integral, where it this time is worthwhile to keep the *abstract* formulation as long as possible.

- 1) On  $\partial K$  we have  $r^2 = x^2 + y^2 + z^2$ , i.e.  $r = (x^2 + y^2 + z^2)^{1/2}$ , and

$$\mathbf{n}(x, y, z) = \frac{1}{r} (x, y, z).$$

- 2) The vector field is now rewritten in the following way

$$\mathbf{V}(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} = \frac{1}{r^3} (x, y, z) = \frac{1}{r^2} \cdot \frac{1}{r} (x, y, z) = \frac{1}{r^2} \mathbf{n},$$

where we have used 1).

3) Since  $\mathbf{n} \cdot \mathbf{n} = \|\mathbf{n}\|^2 = 1$ , we get by an insertion of 2) that the flux is given by

$$\begin{aligned} \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} dS &= \int_{\partial K} \mathbf{V} \cdot \mathbf{n} dS = \int_{\partial K} \frac{1}{r^2} \mathbf{n} \cdot \mathbf{n} dS \\ &= \frac{1}{r^2} \int_{\partial K} dS = \frac{1}{r^2} \text{area}(\partial K) = \frac{1}{r^2} \cdot 4\pi r^2 = 4\pi, \end{aligned}$$

because the area of a sphere of radius  $r$  is given by  $4\pi r^2$ .

The result can be applied in an improved version of the horrible Example 9.6. Let  $\Omega = K(\mathbf{0}; r)$ , where  $r > \sqrt{3}a$ , and let  $T$  be the cube of centre  $\mathbf{0}$  and edge length  $2a$ . Then the flux through  $\partial T$  is equal to the flux through the sphere  $\partial\Omega$ , i.e. according to the above,

$$\int_{\partial T} \mathbf{V} \cdot \mathbf{n} dS = \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} dS = 4\pi.$$

On the other hand,  $\partial T$  is disintegrated in a natural way into six squares of the same congruent form: They appear from each other by a convenient revolution around one of the axis. The Coulomb field is due to its symmetry invariant (apart from a change of letters) by these revolutions, so the flux is the same through every one of the six squares. If we choose one of these. e.g.

$$\mathcal{F} = \{(x, y, z) \mid -a \leq x \leq a, -a \leq y \leq a, z = a\} = [-a, a] \times [-a, a] \times \{a\},$$

then

$$4\pi = \int_{\partial T} \mathbf{V} \cdot \mathbf{n} dS = 6 \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS,$$

from which

$$\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS = \frac{1}{6} \cdot 4\pi = \frac{2\pi}{3}.$$

Obviously this method is far easier in its calculations than the method applied in Example 9.6.  $\diamond$

**Example 9.11** Assume that  $\Omega$  e.g. represents a subsoil water reservoir, which is polluted by some fluid or gas of *density*  $\rho = \rho(x, y, z, t)$  and *velocity vector*  $\mathbf{v} = \mathbf{v}(x, y, z, t)$ .

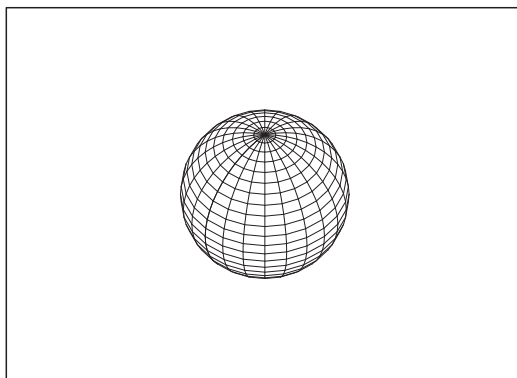


Figure 77: In most calculus courses  $\Omega$  is typically a ball.

The *mass* of the polluting agent in  $\Omega$  at time  $t$  is given by

$$M = M(t) = \int_{\Omega} \varrho(x, y, z, t) d\Omega = \int_{\Omega} \varrho d\Omega.$$

The *change of mass* in time is then obviously equal to

$$(17) \quad \frac{dM}{dt} = \frac{d}{dt} \int_{\Omega} \varrho(x, y, z, t) d\Omega = \int_{\Omega} \frac{\partial \varrho}{\partial t} d\Omega.$$

This change must be equal to the flow of mass *into*  $\Omega$  by the vector field

$$\varrho \mathbf{v} = \varrho(x, y, z, t) \mathbf{v}(x, y, z, t).$$

Since  $-\mathbf{n}$  points *into*  $\Omega$ , this amount is according to *Gauß's theorem* equal to

$$-q = - \int_{\partial\Omega} \varrho \mathbf{v} \cdot \mathbf{n} dS = - \int_{\Omega} \operatorname{div}(\varrho \mathbf{v}) d\Omega.$$

When this expression is equated to (17), we get after a rearrangement that

$$0 = q + \frac{dM}{dt} = \int_{\Omega} \operatorname{div}(\varrho \mathbf{v}) d\Omega + \int_{\Omega} \frac{\partial \varrho}{\partial t} d\Omega = \int_{\Omega} \left\{ \operatorname{div}(\varrho \mathbf{v}) + \frac{\partial \varrho}{\partial t} \right\} d\Omega.$$

This is true for *every* domain  $\Omega$ . Assuming that the integrand is *continuous* (what is always is in practical applications), it must be 0 everywhere. In fact, if the integrand e.g. was positive in a point, then it had due to the continuity also to be positive in an open domain  $\Omega_1$ , and then the integral over  $\Omega_1$  becomes positive too, contradicting the assumption.

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Thus we have derived the *continuity equation*

$$\operatorname{div}(\varrho \mathbf{v}) + \frac{\partial \varrho}{\partial t} = 0,$$

which the density and the velocity vector field of the pollution vector field must satisfy.

**Remark 9.5** Here the divergence is referring to the *spatial* variables and *not* to the time variable  $t$ . thus the continuity equation is written in all details in the following way

$$\frac{\partial}{\partial x} (\varrho v_x) + \frac{\partial}{\partial y} (\varrho v_y) + \frac{\partial}{\partial z} (\varrho v_z) + \frac{\partial \varrho}{\partial t} = 0.$$

Furthermore it should be noted that there is a *big difference* here between the application of  $\frac{d}{dt}$  and  $\frac{\partial}{\partial t}$ .  $\diamond$

### 9.3 Rotation of a vector field; Stokes's theorem

#### Example 9.12

A. Find the circulation  $C$  of the vector field

$$\mathbf{V}(x, y, z) = (z^2x, x^2y, y^2z), \quad (x, y, z) \in \mathbb{R}^3,$$

along the curve  $\mathcal{K}$  on the figure which is composed of three circle arcs of centre  $\mathbf{0}$  and radius  $a$ , and which lie in the planes  $z = 0$ ,  $y = -x$  and  $x = y\sqrt{3}$ , respectively.

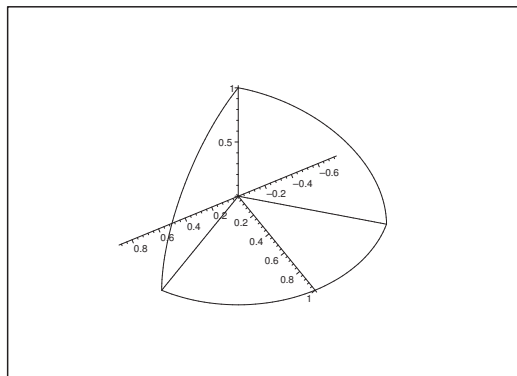


Figure 78: The curve  $\mathcal{K}$  for  $a = 1$ .

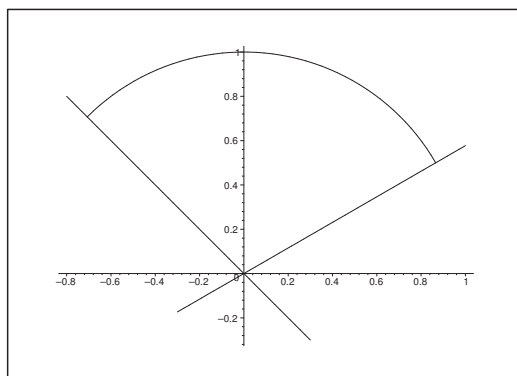


Figure 79: The projection onto the  $(x, y)$ -plane for  $a = 1$ .

**D.** A circulation along a closed curve can either be calculated by using its definition as an ordinary *line integral*, or it can be transformed by means of *Stokes's theorem* to a surface integral. Most students are at their first encounter with this problem inclined to preferring the line integral, because it should now be better known, even when applications of *Stokes's theorem* very often give much simpler calculations. For that reason we shall here demonstrate both variants.

**I 1.** *The circulation as a line integral.*

Let us first give the parametric representations of the three arcs, which  $\mathcal{K}$  can be composed into in a natural way:

$$\begin{aligned}\mathcal{K}_1 : (x, y, z) &= a(\cos \varphi, \sin \varphi, 0), & \varphi &\in \left[ \frac{\pi}{6}, \frac{3\pi}{4} \right], \\ \mathcal{K}_2 : (x, y, z) &= a \left( -\frac{1}{\sqrt{2}} \cos \theta, \frac{1}{\sqrt{2}} \cos \theta, \sin \theta \right), & \theta &\in \left[ 0, \frac{\pi}{2} \right], \\ \mathcal{K}_3 : (x, y, z) &= a \left( \frac{\sqrt{3}}{2} \sin \theta, \frac{1}{2} \sin \theta, \cos \theta \right), & \theta &\in \left[ 0, \frac{\pi}{2} \right].\end{aligned}$$

Check here that all three curves are circular arcs of radius  $a$  and centre  $\mathbf{0}$ . Then check that the initial and the end points are the right ones, i.e. that the direction of the run through is correct. (Here this is left to the reader).

In order to avoid too many complications in the calculations we rewrite the integrand in the following way,

$$\begin{aligned}\mathbf{V} \cdot d\mathbf{x} &= z^2 x dx + x^2 y dy + y^2 z dz \\ &= \frac{1}{2} \{ z^2 d(x^2) + x^2 d(y^2) + y^2 d(z^2) \}.\end{aligned}$$

By insertion we get the following confusing picture of formulæ, where there are lots of possibilities of making errors,

$$\begin{aligned}C &= \frac{1}{2} \int_{\mathcal{K}_1} + \frac{1}{2} \int_{\mathcal{K}_2} + \frac{1}{2} \int_{\mathcal{K}_3} (z^2 d(x^2) + x^2 d(y^2) + y^2 d(z^2)) \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \left\{ 0 \cdot \frac{d(x^2)}{d\varphi} + a^2 \cos^2 \varphi \cdot 2a^2 \sin \varphi \cdot \cos \varphi + y^2 \cdot 0 \right\} d\varphi \\ &\quad + \frac{1}{2} \int_0^{\frac{\pi}{2}} \left\{ a^2 \sin^2 \theta \cdot \frac{a^2}{2} (-2 \cos \theta \sin \theta) + \frac{a^2}{2} \cos^2 \theta \cdot \frac{a^2}{2} (-2 \cos \theta \sin \theta) \right. \\ &\quad \left. + \frac{a^2}{2} \cos^2 \theta \cdot a^2 \cdot 2 \sin \theta \cos \theta \right\} d\theta \\ &\quad + \frac{1}{2} \int_0^{\frac{\pi}{2}} \left\{ a^2 \cos^2 \theta \cdot \frac{3a^2}{4} \cdot 2 \cos \theta \sin \theta + \frac{3a^2}{4} \sin^2 \theta \cdot \frac{a^2}{4} \cdot 2 \cos \theta \sin \theta \right. \\ &\quad \left. + \frac{a^2}{4} \sin^2 \theta \cdot a^2 (-2 \sin \theta \cos \theta) \right\} d\theta \\ &= a^4 \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cos^3 \varphi \sin \varphi d\varphi + \frac{a^4}{2} \int_0^{\frac{\pi}{2}} \left( -\sin^3 \theta \cos \theta - \frac{1}{2} \cos^3 \theta \sin \theta + \cos^3 \theta \sin \theta \right) d\theta \\ &\quad + \frac{a^4}{2} \int_0^{\frac{\pi}{2}} \left\{ \frac{3}{2} \cos^3 \theta \sin \theta + \frac{3}{8} \sin^3 \theta \cos \theta - \frac{1}{2} \sin^3 \theta \cos \theta \right\} d\theta,\end{aligned}$$

i.e.

$$\begin{aligned}
 C &= a^4 \left[ -\frac{1}{4} \cos^4 \varphi \right]_{\frac{\pi}{6}}^{\frac{3\pi}{4}} + \frac{a^4}{2} \int_0^{\frac{\pi}{2}} \left\{ \left( -1 + \frac{3}{8} - \frac{1}{2} \right) \sin^3 \theta \cos \theta \right. \\
 &\quad \left. + \left( -\frac{1}{2} + 1 + \frac{3}{2} \right) \cos^3 \theta \sin \theta \right\} d\theta \\
 &= \frac{a^4}{4} \left\{ \left( \frac{\sqrt{3}}{2} \right)^4 - \left( -\frac{1}{\sqrt{2}} \right)^4 \right\} + \frac{a^4}{2} \int_0^{\frac{\pi}{2}} \left\{ -\frac{9}{8} \sin^3 \theta \cos \theta + 2 \cos^3 \theta \sin \theta \right\} d\theta \\
 &= \frac{a^4}{4} \left( \frac{9}{16} - \frac{1}{4} \right) + \frac{a^4}{2} \left[ -\frac{9}{8} \cdot \frac{1}{4} \sin^4 \theta - 2 \cdot \frac{1}{4} \cos^4 \theta \right]_0^{\frac{\pi}{2}} \\
 &= \frac{5a^4}{64} + \frac{a^4}{2} \left( -\frac{9}{32} - 0 + 0 + \frac{1}{2} \right) = \frac{a^4}{64} (5 - 9 + 16) = \frac{12a^4}{64} = \frac{3a^4}{16}.
 \end{aligned}$$

The calculations can be carried through, though they are far from simple.

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**I 2.** *Circulation by means of Stokes's theorem.*

We shall now demonstrate that the calculations by an application of Stokes's theorem in this case is far simpler. First note that since all sub-curves are circular arcs of radius  $a$  and centre  $\mathbf{0}$ , they all lie on the sphere of radius  $a$  and centre  $\mathbf{0}$ . They bound a part  $\mathcal{F}$  of the sphere, where the orientation of  $\mathcal{K}$  forces the normal vector  $\mathbf{n}$  on  $\mathcal{F}$  to point away from  $\mathbf{0}$  (the right hand convention), i.e.

$$\mathbf{n} = \frac{1}{a} (x, y, z), \quad \text{where } a = \sqrt{x^2 + y^2 + z^2}.$$

Furthermore,

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2x & x^2y & y^2z \end{vmatrix} = (2yz, 2zx, 2xy).$$

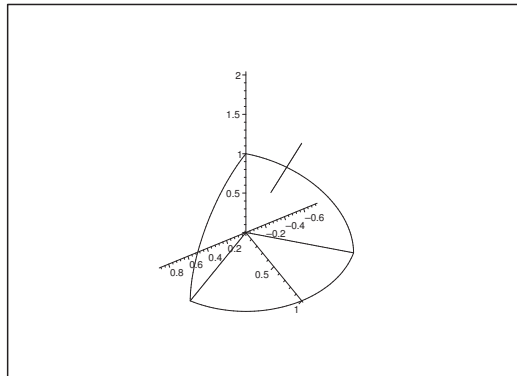


Figure 80: The surface  $\mathcal{F}$  for  $a = 1$  with a single normal vector  $\mathbf{n}$ .

It is on the sphere most natural to use *spherical* coordinates. Then by Stokes's theorem and a reduction,

$$\begin{aligned} C &= \int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} = \int_{\mathcal{F}} \mathbf{n} \cdot \text{rot } \mathbf{V} \, dS \\ &= \int_{\mathcal{F}} \frac{1}{a} (x, y, z) \cdot (2yz, 2zx, 2xy) \, dS = \frac{6}{a} \int_{\mathcal{F}} xyz \, dS \\ &= \frac{6}{a} \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \left\{ \int_0^{\frac{\pi}{2}} a \sin \theta \cos \varphi \cdot a \sin \theta \sin \varphi \cdot a \cos \theta a^2 \sin \theta \, d\theta \right\} d\varphi \\ &= 6a^4 \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cos \varphi \cdot \sin \varphi \, d\varphi \cdot \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos \theta \, d\theta \\ &= 6a^4 \left[ \frac{1}{2} \sin^2 \varphi \right]_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cdot \left[ \frac{1}{4} \sin^4 \theta \right]_0^{\frac{\pi}{2}} \\ &= 6a^4 \cdot \frac{1}{2} \left\{ \frac{1}{2} - \frac{1}{4} \right\} \cdot \frac{1}{4} = \frac{6a^4}{32} = \frac{3a^4}{16}, \end{aligned}$$

i.e. far easier calculations than by the line integral.  $\diamond$

**Example 9.13**

A. An oriented curve  $\mathcal{K}$  is given as the intersection curve between

- the cylindric surface  $x^2 + y^2 = ax$ ,

and

- half of the parabolic cylindric surface

$$z = \sqrt{4a^2 - ax}.$$

The direction of the run through of  $\mathcal{K}$  obeys the right hand convention with the positive  $z$ -axis.

Find the circulation along  $\mathcal{K}$  of the vector field

$$\mathbf{V}(x, y, z) = (3xy, 2x^2, -yz), \quad (x, y, z) \in \mathbb{R}^3.$$

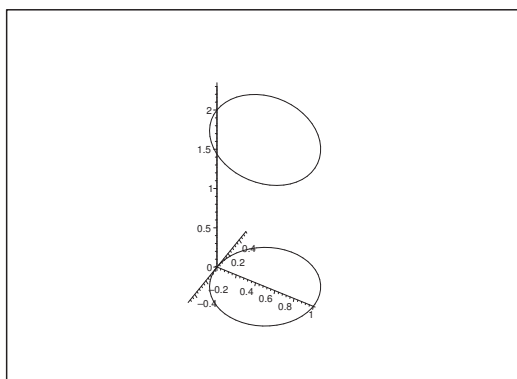


Figure 81: The curve  $\mathcal{K}$  and its projection onto the  $(x, y)$ -plane.

D. We shall here demonstrate *three* variants:

- 1) *The line integral* (the calculations are almost worse than those of Example 9.12).
- 2) *Stokes's theorem* by the surface

$$\mathcal{F}_1: \quad z = \sqrt{4a^2 - ax}, \quad (x, y) \in B,$$

- 3) *Stokes' sætning* by the surface

$$\mathcal{F}_2: \quad x^2 + y^2 + z^2 = (2a)^2, \quad (x, y) \in B, \quad z > 0.$$

I 1. As a general warning we shall first bring the solution calculated as a *line integral*.

When  $\mathcal{K}$  is projected onto the  $(x, y)$ -plane we get the curve  $\mathcal{L}$  of the parametric representation

$$(x, y, z) = (\varrho \cos \varphi, \varrho \sin \varphi, 0) = a (\cos^2 \varphi, \cos \varphi \sin \varphi, 0), \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Now  $z = \sqrt{4a^2 - ax}$ , so  $\mathcal{K}$  has the parametric representation

$$\mathcal{K}: \quad \mathbf{r}(\varphi) = (x, y, z) = a (\cos^2 \varphi, \cos \varphi \sin \varphi, \sqrt{4 - \cos^2 \varphi}), \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Then we get the values of the vector field along the curve  $\mathcal{K}$ ,

$$\begin{aligned} \mathbf{V}(x, y, z) &= (3xy, 2x^2, -yz) \\ &= \left(3a \cos^2 \varphi \cdot a \cos \varphi \sin \varphi, 2a^2 \cos^4 \varphi, -a \cos \varphi \sin \varphi \cdot a \sqrt{4 - \cos^2 \varphi}\right) \\ &= a^2 \left(3 \cos^3 \varphi \sin \varphi, 2 \cos^4 \varphi, -\cos \varphi \cdot \sin \varphi \cdot \sqrt{4 - \cos^2 \varphi}\right). \end{aligned}$$

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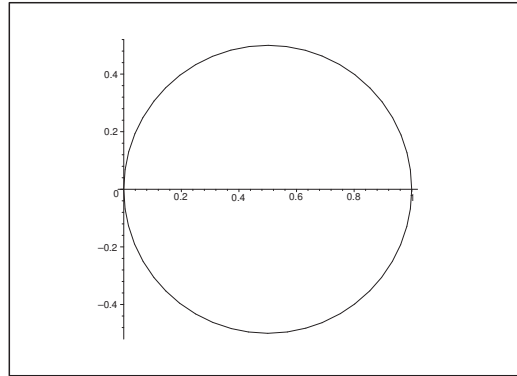


Figure 82: The parametric domain  $B$  is the disc in the  $(x, y)$ -plane of centre  $\left(\frac{a}{2}, 0\right)$  and radius  $\frac{a}{2}$ . Here we have chosen  $a = 1$ .

Hence,

$$\mathbf{r}'(\varphi) = a \left( -2 \sin \varphi \cdot \cos \varphi, \cos^2 \varphi - \sin^2 \varphi, \frac{\cos \varphi \cdot \sin \varphi}{\sqrt{4 - \cos^2 \varphi}} \right).$$

In order to avoid too big calculations we first calculate the *integrand* separately,

$$\begin{aligned} \mathbf{V} \cdot \mathbf{r}'(\varphi) &= a^3 \left( 3 \cos^3 \varphi \sin \varphi \cdot (-2 \sin \varphi \cos \varphi) + 2 \cos^4 \varphi \cdot (\cos^2 \varphi - \sin^2 \varphi) - \cos^2 \varphi \sin^2 \varphi \right) \\ &= a^3 \left( 3 \cos^2 \varphi \cos \varphi \sin \varphi \cdot (-\sin 2\varphi) + 2(\cos^2 \varphi)^2 \cdot \cos 2\varphi - \frac{1}{4} (2 \cos \varphi \sin \varphi)^2 \right) \\ &= a^3 \left\{ 3 \frac{1 + \cos 2\varphi}{2} \frac{\sin 2\varphi}{2} \cdot (-\sin 2\varphi) + 2 \left( \frac{1 + \cos 2\varphi}{2} \right)^2 \cos 2\varphi - \frac{1}{4} \sin^2 2\varphi \right\} \\ &= \frac{a^3}{4} \left\{ -3(1 + \cos 2\varphi) \sin^2 2\varphi + 2(1 + \cos 2\varphi)^2 \cos 2\varphi - \sin^2 2\varphi \right\} \\ &= \frac{a^3}{4} \left\{ -3 \sin^2 2\varphi - 3 \cos 2\varphi \sin^2 2\varphi + 2(1 + 2 \cos 2\varphi + \cos^2 2\varphi) \cos 2\varphi - \sin^2 2\varphi \right\} \\ &= \frac{a^3}{4} \left\{ -4 \sin^2 2\varphi - 3 \cos 2\varphi \sin^2 2\varphi + 2 \cos 2\varphi + 4 \cos^2 2\varphi + 2 \cos^2 2\varphi \cdot \cos 2\varphi \right\} \\ &= \frac{a^3}{4} \left\{ 4 (\cos^2 2\varphi - \sin^2 2\varphi) - 5 \cos 2\varphi \sin^2 2\varphi + 2 \cos 2\varphi (\sin^2 2\varphi + \cos^2 2\varphi) + 2 \cos 2\varphi \right\}, \end{aligned}$$

i.e.

$$\mathbf{V} \cdot \mathbf{r}'(\varphi) = \frac{a^3}{4} \left\{ 4 \cos 4\varphi - 5 \cos 2\varphi \sin^2 2\varphi + 4 \cos 2\varphi \right\},$$

which is a fairly tough calculation. At the same time we see why we do not immediately insert the expression into the integral.

However, after this tough reduction the circulation becomes easy to calculate,

$$\begin{aligned}
 C &= \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{V} \cdot \mathbf{r}'(\varphi) \, d\varphi \\
 &= \frac{a^3}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{4 \cos 4\varphi - 5 \cos 2\varphi \sin^2 2\varphi + 4 \cos 2\varphi\} \, d\varphi \\
 &= \frac{a^3}{4} \left\{ [\sin 4\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 5 \left[ \frac{1}{2} \cdot \frac{1}{3} \sin^3 2\varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + [2 \sin 2\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right\} \\
 &= 0,
 \end{aligned}$$

and we see that after so much trouble we only get zero as our result.

**I.** As an introduction to the applications of *Stokes's theorem* we first calculate

$$\mathbf{rot} \, \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & 2x^2 & -yz \end{vmatrix} = (-z, 0, 4x - 3x) = (-z, 0, x),$$

which looks promising, because  $\mathbf{rot} \, \mathbf{V}$  is much simpler than  $\mathbf{V}$ .

Then we shall choose a surface which has  $\mathcal{K}$  as its boundary curve. This surface is of course not unique.

**I 2.** The most obvious possibility is to choose the parabolic cylindric surface

$$\mathcal{F}_1 : \quad z = \sqrt{4a^2 - ax}, \quad (x, y) \in B,$$

where the parametric domain  $B$  is best described in *polar* coordinates,

$$B = \left\{ (\varrho, \varphi) \mid \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], 0 \leq \varrho \leq a \cos \varphi \right\},$$

while  $(x, y, z)$  still denotes the *rectangular* coordinates.

Because  $\mathcal{F}_1$  is given as a *graph*, the parametric representation is

$$\mathbf{r}(x, y) = (x, y, z) = \left(x, y, \sqrt{4a^2 - ax}\right), \quad (x, y) \in B,$$

from which we get the tangential vector fields

$$\mathbf{r}'_x = \left(1, 0, \frac{1}{2} \frac{-a}{\sqrt{4a^2 - ax}}\right) = \left(1, 0, -\frac{a}{2z}\right), \quad \text{og} \quad \mathbf{r}'_y = (0, 1, 0),$$

and the corresponding normal vector field

$$\mathbf{N}(x, y) = \mathbf{r}'_x \times \mathbf{r}'_y = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{a}{2z} \\ 0 & 1 & 0 \end{vmatrix} = \left(\frac{a}{2z}, 0, 1\right).$$

We see that  $\mathbf{N}(x, y)$  and the orientation of  $\mathcal{K}$  satisfy the right hand convention  $\mathbf{N} \cdot \mathbf{e}_z = 1$ , so we have the correct orientation.

An application of *Stokes's theorem* now gives

$$\begin{aligned}
 C &= \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}_1} \mathbf{n} \cdot \mathbf{rot} \, \mathbf{V} \, dS = \int_B \left( \frac{a}{2z}, 0, 1 \right) \cdot (-z, 0, x) \, dx \, dy \\
 &= \int_B \left( -\frac{1}{2}a + x \right) \, dx \, dy = -\frac{a}{2} \text{area}(B) + \int_B x \, dx \, dy \\
 &= -\frac{a}{2} \cdot \pi \left( \frac{a}{2} \right)^2 + \int_{-\frac{\pi}{2}}^{\pi} 2 \left\{ \int_0^{a \cos \varphi} \varrho \cos \varphi \cdot \varrho \, d\varrho \right\} d\varphi \\
 &= -\frac{\pi a^3}{8} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi \cdot \left[ \frac{1}{3} \varrho^3 \right]_0^{a \cos \varphi} d\varphi = -\frac{\pi a^3}{8} + \frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \varphi \, d\varphi \\
 &= -\frac{\pi a^3}{8} + \frac{2a^3}{3} \int_0^{\frac{\pi}{2}} \left( \frac{1 + \cos 2\varphi}{2} \right)^2 d\varphi \\
 &= -\frac{\pi a^3}{8} + \frac{2a^3}{3} \cdot \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 + 2 \cos 2\varphi + \cos^2 2\varphi) \, d\varphi \\
 &= -\frac{\pi a^3}{8} + \frac{a^3}{6} \left\{ \frac{\pi}{2} + 0 + \frac{\pi}{4} \right\} = 0.
 \end{aligned}$$

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**I 3.** The calculations of **I 2.** were much simpler than the calculations of **I 1.**, although they still gave some trigonometric problems. The question is now, if it is possible to choose another surface  $\mathcal{F}$  with  $\mathcal{K}$  as its boundary curve, such that the calculations become even more easy. We shall now show that this is possible, though far from obvious.

First we note that since  $K$  satisfies the two conditions

$$\text{a) } x^2 + y^2 = ax \quad \text{and} \quad \text{b) } z^2 = 4a^2 - ax,$$

the parametric representation of  $\mathcal{F}$  must also satisfy

$$(x^2 + y^2) + z^2 = ax + (4a^2 - ax) = 4a^2 = (2a)^2, \quad z > 0,$$

i.e.  $\mathcal{K}$  lies on the half sphere of centre  $\mathbf{0}$  and radius  $2a$  and  $z > 0$ . We choose  $\mathcal{F}_2$  as that part of this half sphere which lies above the parametric domain  $B$ :

$$\mathcal{F}_2: \quad x^2 + y^2 + z^2 = (2a)^2, \quad (x, y) \in B, \quad z \geq 0.$$

The sphere of centre  $\mathbf{0}$  and radius  $2a$  has the unit normal vector

$$\mathbf{n} = \frac{1}{2a} (x, y, z),$$

hence we have on  $\mathcal{F}_2$

$$\mathbf{n} \cdot \mathbf{rot} \mathbf{V} = \frac{1}{2a} (x, y, z) \cdot (-z, 0, x) = 0.$$

By an application of *Stokes's theorem* the calculation of the circulation is now reduced to a triviality,

$$C = \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}_2} \mathbf{n} \cdot \mathbf{rot} \mathbf{V} \, dS = \int_{\mathcal{F}_2} 0 \, dS = 0,$$

in which we shall not even insert a parametric representation followed by some reduction theorem!  
 $\diamond$

## 9.4 Potentials

### Example 9.14

**A.** Given the divergence free vector field

$$\mathbf{V}(x, y, z) = (3y^2z^2, xy, -xz), \quad (x, y, z) \in \mathbb{R}^3.$$

Since  $\mathbb{R}^3$  is star-shaped,  $\mathbf{V}$  has vector potentials  $\mathbf{W}$ . Find one of these, i.e. find  $\mathbf{W}$ , such that

$$\mathbf{V} = \nabla \times \mathbf{W} = \mathbf{rot} \mathbf{W}.$$

**D.** The usual solution formula of textbooks looks here very harmless, but from my experience of teaching I can say that it is *not* that harmless. Therefore we shall give a thorough treatment of this example.

The method can briefly be described in the following way:

- 1) Have this exercise been solved earlier in the text?
- 2) If 'no', examine if  $\mathbf{V}(x, y, z)$  is divergence free.
- 3) Calculate separately the auxiliary field  $\mathbf{T}(\mathbf{x}) = \mathbf{V}(\mathbf{x}) \times \mathbf{x}$ .
- 4) Replace  $\mathbf{x}$  by  $\tau \mathbf{x}$ . (This is the critical phase of the method).
- 5) Calculate  $\mathbf{W}_0(\mathbf{x}) = \int_0^1 \mathbf{T}(\tau \mathbf{x}) d\tau$ .
- 6) Check once more the preassumptions of the method.
- 7) Check the result.

I. 1) *We first check if we earlier have solve the problem.*

It happens quite often at an examination that one in the first part of a problem as a check are asked to calculate the rotation of e.g. the vector field

$$\mathbf{W}_1(x, y, z) = \left( xyz, -\frac{1}{2}y^2z^3, \frac{1}{2}y^3z^2 \right).$$

This is of course given by

$$\begin{aligned} \nabla \times \mathbf{W}_1 &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -\frac{1}{2}y^2z^3 & \frac{1}{2}y^3z^2 \end{vmatrix} \\ &= \left( \frac{3}{2}y^2z^2 + \frac{3}{2}y^2z^2, xy - 0, 0 - xz \right) = (3y^2z^2, xy, -xz) = \mathbf{V}. \end{aligned}$$

Half an hour later one comes to another question in which one is asked to find a vector potential for  $\mathbf{V}$ . Some students may immediately see that  $\mathbf{W}_1$  is a solution. Other poor souls have to go through the following steps:

- 2) *Examine if  $\mathbf{V}(x, y, z)$  is divergence free.*

If not, there is no vector potential, and the problem is solved.

In the example under consideration we get

$$\operatorname{div} \mathbf{V} = 0 + x - x = 0,$$

so we must carry on to the next point.

- 3) *Calculate the auxiliary field  $\mathbf{T}(\mathbf{x}) = \mathbf{V}(\mathbf{x}) \times \mathbf{x}$ .*

Be very careful with the order of the factors. And do not yet introduce the parameter  $\tau$  in the calculations. It will only confuse the overview.

In the case under consideration we get

$$\begin{aligned} \mathbf{T}(\mathbf{x}) &= \mathbf{V}(\mathbf{x}) \times \mathbf{x} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 3y^2z^2 & xy & -xz \\ x & y & z \end{vmatrix} \\ &= (xyz - (-xyz), -x^2z3y^2z^3, 3y^3z^2 - x^2y) \\ &= (2xyz, -x^2z - 3y^2z^3, 3y^3z^2 - x^2y). \end{aligned}$$



4) Replace  $x$  by  $\tau x$ ,  $y$  by  $\tau y$  and  $z$  by  $\tau z$ .

This is the critical step where most errors are made. Be extremely scrupulous here. It may be useful to note that one in polynomials just add the factor  $\tau^n$  to every term which contains  $n$  factors of the type  $x$ ,  $y$  or  $z$  (they are considered as equal at this count).

In the case under consideration where  $\mathbf{T}(\mathbf{x})$  is composed of polynomials we get

$$\mathbf{T}(\tau \mathbf{x}) = (2\tau^3 xyz, -\tau^3 x^2 z - 3\tau^5 y^2 z^3, 3\tau^5 y^3 z^2 - t^4 x^2 y),$$

because  $xyz$ ,  $x^2 z$  and  $x^2 y$  contain three factors, i.e. we shall multiply by  $\tau^3$ , while  $y^2 z^3$  and  $y^3 z^2$  contain five factors, so here we add the factor  $\tau^5$ .

5) Calculate the candidate

$$\mathbf{W}_0(\mathbf{x}) = \int_0^1 \mathbf{T}(\tau \mathbf{x}) d\tau$$

by a coordinate-wise integration after  $\tau$ .

We have in the case under consideration

$$\begin{aligned} \mathbf{W}_0(\mathbf{x}) &= \int_0^1 \mathbf{T}(\tau \mathbf{x}) d\tau \\ &= \left( xyz \int_0^1 2\tau^3 d\tau, -x^2 z \int_0^1 \tau^3 d\tau - y^2 z^3 \int_0^1 3\tau^5 d\tau, y^3 z^2 \int_0^1 3\tau^5 d\tau - x^2 y \int_0^1 \tau^3 d\tau \right) \\ &= \left( \frac{1}{2} xyz, -\frac{1}{4} x^2 z - \frac{1}{2} y^2 z^3, \frac{1}{2} y^3 z^2 - \frac{1}{4} x^2 y \right). \end{aligned}$$

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The expressions are often so huge that it pays to calculate each coordinate separately. We see that even in this simple case the equations are enormous.

- 6) If our calculations have been correct – notice that  $\mathbf{0}$  must lie in the domain – then  $\mathbf{W}_0(\mathbf{x})$  is a vector potential for  $\mathbf{V}(\mathbf{x})$  in every sub-domain of  $\Omega$ , which can be reached by a radial line in  $\Omega$  from  $\mathbf{0}$ .

In the case under consideration we see that  $\mathbb{R}^3$  satisfies this criterium, hence

$$\mathbf{W}_0(\mathbf{x}) = \left( \frac{1}{2}xyz, -\frac{1}{4}x^2z - \frac{1}{2}y^2z^3, \frac{1}{2}y^3z^2 - \frac{1}{4}x^2y \right)$$

is a vector potential in the whole of  $\mathbb{R}^3$  (if our calculations were without errors, of course).

- 7) By a comparison we see that we have obtained two different solutions,  $\mathbf{W}_0(\mathbf{x})$  in the latter case and  $\mathbf{W}_1(\mathbf{x})$  in the former one. They are both correct, because their difference is only a gradient field. Nevertheless this phenomenon often causes some panic after the examination, because students are used to from high school that the solution is unique. It is *not* in problems of this type!

Since the method is so difficult it is highly *recommended that one always check every calculation in exercises of this type*. In the present case we get

$$\begin{aligned} \nabla \times \mathbf{W}_0 &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}xyz & -\frac{1}{4}x^2z - \frac{1}{2}y^2z^3 & \frac{1}{2}y^3z^2 - \frac{1}{4}x^2y \end{vmatrix} \\ &= \left( \frac{3}{2}y^2z^2 - \frac{1}{4}x^2 - \left\{ -\frac{1}{4}x^2 - \frac{3}{2}y^2z^2 \right\}, \frac{1}{2}xy - \left\{ 0 - \frac{1}{2}xy \right\}, -\frac{1}{2}xz - \frac{1}{2}xz \right) \\ &= (3y^2z^2, xy, -xz) = \mathbf{V}(x, y, z). \end{aligned}$$

We have now checked that  $\mathbf{W}_0$  is a vector potential.

**Remark 9.6** If a check does not give the right answer and  $\text{div } \mathbf{V} = 0$ , then we have made an error at some place, probably at 4) which would be the first place where I would search myself. This does not leave out the possibility of errors in the other steps as well.  $\diamond$

## A Formulæ

Some of the following formulæ can be assumed to be known from high school. Others are introduced in Calculus 1. It is highly recommended that one *learns most of these formulæ in this appendix by heart*.

### A.1 Squares etc.

The following simple formulæ occurs very frequently in the most different situations.

$$\begin{array}{ll} (a+b)^2 = a^2 + b^2 + 2ab, & a^2 + b^2 + 2ab = (a+b)^2, \\ (a-b)^2 = a^2 + b^2 - 2ab, & a^2 + b^2 - 2ab = (a-b)^2, \\ (a+b)(a-b) = a^2 - b^2, & a^2 - b^2 = (a+b)(a-b), \\ (a+b)^2 = (a-b)^2 + 4ab, & (a-b)^2 = (a+b)^2 - 4ab. \end{array}$$

### A.2 Powers etc.

**Logarithm:**

$$\begin{array}{ll} \ln |xy| = \ln |x| + \ln |y|, & x, y \neq 0, \\ \ln \left| \frac{x}{y} \right| = \ln |x| - \ln |y|, & x, y \neq 0, \\ \ln |x^r| = r \ln |x|, & x \neq 0. \end{array}$$

**Power function, fixed exponent:**

$$\begin{array}{ll} (xy)^r = x^r \cdot y^r, x, y > 0 & \text{(extensions for some } r), \\ \left( \frac{x}{y} \right)^r = \frac{x^r}{y^r}, x, y > 0 & \text{(extensions for some } r). \end{array}$$

**Exponential, fixed base:**

$$\begin{array}{ll} a^x \cdot a^y = a^{x+y}, a > 0 & \text{(extensions for some } x, y), \\ (a^x)^y = a^{xy}, a > 0 & \text{(extensions for some } x, y), \\ a^{-x} = \frac{1}{a^x}, a > 0, & \text{(extensions for some } x), \\ \sqrt[n]{a} = a^{1/n}, a \geq 0, & n \in \mathbb{N}. \end{array}$$

**Square root:**

$$\sqrt{x^2} = |x|, \quad x \in \mathbb{R}.$$

**Remark A.1** It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: “If you can master the square root, you can master everything in mathematics!” Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!*  $\diamond$

### A.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x) \left\{ \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \right\},$$

where the latter rearrangement presupposes that  $f(x) \neq 0$  and  $g(x) \neq 0$ . If  $g(x) \neq 0$ , we get the usual formula known from high school

$$\left\{ \frac{f(x)}{g(x)} \right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{ \frac{f(x)}{g(x)} \right\}' = \frac{d}{dx} \left\{ f(x) \cdot \frac{1}{g(x)} \right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)} \left\{ \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that  $f(x) \neq 0$  and  $g(x) \neq 0$ . Under these assumptions we see that the formulæ above can be written

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$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx} \ln |f(x)| = \frac{f'(x)}{f(x)}, \quad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

## A.4 Special derivatives.

**Power like:**

$$\frac{d}{dx} (x^\alpha) = \alpha \cdot x^{\alpha-1}, \quad \text{for } x > 0, \text{ (extensions for some } \alpha).$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad \text{for } x \neq 0.$$

**Exponential like:**

$$\frac{d}{dx} \exp x = \exp x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} (a^x) = \ln a \cdot a^x, \quad \text{for } x \in \mathbb{R} \text{ og } a > 0.$$

**Trigonometric:**

$$\frac{d}{dx} \sin x = \cos x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \cos x = -\sin x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx} \cot x = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, \quad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

**Hyperbolic:**

$$\frac{d}{dx} \sinh x = \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \cosh x = \sinh x, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \quad \text{for } x \neq 0.$$

**Inverse trigonometric:**

$$\frac{d}{dx} \operatorname{Arcsin} x = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } x \in ]-1, 1[,$$

$$\frac{d}{dx} \operatorname{Arccos} x = -\frac{1}{\sqrt{1-x^2}}, \quad \text{for } x \in ]-1, 1[,$$

$$\frac{d}{dx} \operatorname{Arctan} x = \frac{1}{1+x^2}, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \operatorname{Arccot} x = \frac{1}{1+x^2}, \quad \text{for } x \in \mathbb{R}.$$

**Inverse hyperbolic:**

$$\frac{d}{dx} \operatorname{Arsinh} x = \frac{1}{\sqrt{x^2+1}}, \quad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx} \operatorname{Arcosh} x = \frac{1}{\sqrt{x^2-1}}, \quad \text{for } x \in ]1, +\infty[,$$

$$\frac{d}{dx} \operatorname{Artanh} x = \frac{1}{1-x^2}, \quad \text{for } |x| < 1,$$

$$\frac{d}{dx} \operatorname{Arcoth} x = \frac{1}{1-x^2}, \quad \text{for } |x| > 1.$$

**Remark A.2** The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class.  $\diamond$

**A.5 Integration**

The most obvious rules are about linearity

$$\int \{f(x) + \lambda g(x)\} dx = \int f(x) dx + \lambda \int g(x) dx, \quad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and about that differentiation and integration are “inverses to each other”, i.e. modulo some arbitrary constant  $c \in \mathbb{R}$ , which often tacitly is missing,

$$\int f'(x) dx = f(x).$$

If we in the latter formula replace  $f(x)$  by the product  $f(x)g(x)$ , we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}' dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx.$$

Hence, by a rearrangement

**The rule of partial integration:**

$$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term  $f(x)g(x)$ .

**Remark A.3** This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself.  $\diamond$

**Remark A.4** This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. See also Chapter 4.  $\diamond$

**Integration by substitution:**

If the integrand has the special structure  $f(\varphi(x)) \cdot \varphi'(x)$ , then one can change the variable to  $y = \varphi(x)$ :

$$\int f(\varphi(x)) \cdot \varphi'(x) dx = \int f(\varphi(x)) d\varphi(x) = \int_{y=\varphi(x)} f(y) dy.$$

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**Integration by a monotonous substitution:**

If  $\varphi(y)$  is a *monotonous* function, which maps the  $y$ -interval *one-to-one* onto the  $x$ -interval, then

$$\int f(x) dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y) dy.$$

**Remark A.5** This rule is usually used when we have some “ugly” term in the integrand  $f(x)$ . The idea is to put this ugly term equal to  $y = \varphi^{-1}(x)$ . When e.g.  $x$  occurs in  $f(x)$  in the form  $\sqrt{x}$ , we put  $y = \varphi^{-1}(x) = \sqrt{x}$ , hence  $x = \varphi(y) = y^2$  og  $\varphi'(y) = 2y$ .  $\diamond$

**A.6 Special antiderivatives**

**Power like:**

$$\int \frac{1}{x} dx = \ln|x|, \quad \text{for } x \neq 0. \text{ (Do not forget the numerical value!)}$$

$$\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1}, \quad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2} dx = \text{Arctan } x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2} dx = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|, \quad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2} dx = \text{Artanh } x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2} dx = \text{Arcoth } x, \quad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \text{Arcsin } x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = -\text{Arccos } x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \text{Arsinh } x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \ln(x + \sqrt{x^2+1}), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \sqrt{x^2-1}, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \text{Arcosh } x, \quad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \ln|x + \sqrt{x^2-1}|, \quad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that  $\sqrt{x^2-1} < |x|$  so if  $x < -1$ , then  $x + \sqrt{x^2-1} < 0$ . Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.



**Exponential like:**

$$\int \exp x \, dx = \exp x, \quad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \quad \text{for } x \in \mathbb{R}, \text{ og } a > 0, a \neq 1.$$

**Trigonometric:**

$$\int \sin x \, dx = -\cos x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln |\cos x|, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln |\sin x|, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left( \frac{1 + \sin x}{1 - \sin x} \right), \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left( \frac{1 - \cos x}{1 + \cos x} \right), \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

**Hyperbolic:**

$$\int \sinh x \, dx = \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln |\sinh x|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln \left( \frac{\cosh x - 1}{\cosh x + 1} \right), \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x} dx = \ln \left| \frac{e^x - 1}{e^x + 1} \right|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x} dx = \tanh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x} dx = -\operatorname{coth} x, \quad \text{for } x \neq 0.$$

### A.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus  $(\cos u, \sin u)$  are the coordinates of a point  $P$  on the unit circle corresponding to the angle  $u$ , cf. figure A.1. This geometrical interpretation is used from time to time.

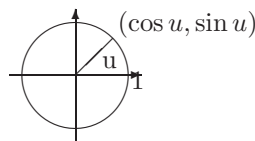


Figure 83: The unit circle and the trigonometric functions.

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**The fundamental trigonometric relation:**

$$\cos^2 u + \sin^2 u = 1, \quad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point  $P$  with the coordinates  $(\cos u, \sin u)$  always has distance 1 from the origo  $(0, 0)$ , i.e. it is lying on the boundary of the circle of centre  $(0, 0)$  and radius  $\sqrt{1} = 1$ .

**Connection to the complex exponential function:**

The *complex exponential* is for imaginary arguments defined by

$$\exp(iu) := \cos u + i \sin u.$$

It can be checked that the usual functional equation for  $\exp$  is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for  $\exp(iu)$  and  $\exp(-iu)$  it is easily seen that

$$\cos u = \frac{1}{2}(\exp(iu) + \exp(-iu)),$$

$$\sin u = \frac{1}{2i}(\exp(iu) - \exp(-iu)),$$

**Moivre's formula:** By expressing  $\exp(inu)$  in two different ways we get:

$$\exp(inu) = \cos nu + i \sin nu = (\cos u + i \sin u)^n.$$

**Example A.1** If we e.g. put  $n = 3$  into Moivre's formula, we obtain the following typical application,

$$\begin{aligned} \cos(3u) + i \sin(3u) &= (\cos u + i \sin u)^3 \\ &= \cos^3 u + 3i \cos^2 u \cdot \sin u + 3i^2 \cos u \cdot \sin^2 u + i^3 \sin^3 u \\ &= \{\cos^3 u - 3 \cos u \cdot \sin^2 u\} + i\{3 \cos^2 u \cdot \sin u - \sin^3 u\} \\ &= \{4 \cos^3 u - 3 \cos u\} + i\{3 \sin u - 4 \sin^3 u\} \end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4 \cos^3 u - 3 \cos u, \quad \sin 3u = 3 \sin u - 4 \sin^3 u. \quad \diamond$$

**Addition formulæ:**

$$\sin(u + v) = \sin u \cos v + \cos u \sin v,$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v,$$

$$\cos(u + v) = \cos u \cos v - \sin u \sin v,$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v.$$

**Products of trigonometric functions to a sum:**

$$\sin u \cos v = \frac{1}{2} \sin(u + v) + \frac{1}{2} \sin(u - v),$$

$$\cos u \sin v = \frac{1}{2} \sin(u + v) - \frac{1}{2} \sin(u - v),$$

$$\sin u \sin v = \frac{1}{2} \cos(u - v) - \frac{1}{2} \cos(u + v),$$

$$\cos u \cos v = \frac{1}{2} \cos(u - v) + \frac{1}{2} \cos(u + v).$$

**Sums of trigonometric functions to a product:**

$$\sin u + \sin v = 2 \sin \left( \frac{u+v}{2} \right) \cos \left( \frac{u-v}{2} \right),$$

$$\sin u - \sin v = 2 \cos \left( \frac{u+v}{2} \right) \sin \left( \frac{u-v}{2} \right),$$

$$\cos u + \cos v = 2 \cos \left( \frac{u+v}{2} \right) \cos \left( \frac{u-v}{2} \right),$$

$$\cos u - \cos v = -2 \sin \left( \frac{u+v}{2} \right) \sin \left( \frac{u-v}{2} \right).$$

**Formulæ of halving and doubling the angle:**

$$\sin 2u = 2 \sin u \cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2 \cos^2 u - 1 = 1 - 2 \sin^2 u,$$

$$\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

$$\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

**A.8 Hyperbolic formulæ**

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

**The fundamental relation:**

$$\cosh^2 x - \sinh^2 x = 1.$$

**Definitions:**

$$\cosh x = \frac{1}{2} (\exp(x) + \exp(-x)), \quad \sinh x = \frac{1}{2} (\exp(x) - \exp(-x)).$$

**“Moivre’s formula”:**

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

**Addition formulæ:**

$$\sinh(x + y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y),$$

$$\sinh(x - y) = \sinh(x) \cosh(y) - \cosh(x) \sinh(y),$$

$$\cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y),$$

$$\cosh(x - y) = \cosh(x) \cosh(y) - \sinh(x) \sinh(y).$$

**Formulæ of halving and doubling the argument:**

$$\sinh(2x) = 2 \sinh(x) \cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2 \cosh^2(x) - 1 = 2 \sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm \sqrt{\frac{\cosh(x) - 1}{2}} \quad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x) + 1}{2}}.$$

**Inverse hyperbolic functions:**

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right), \quad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1,$$

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$$\operatorname{Artanh}(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right), \quad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2} \ln \left( \frac{x+1}{x-1} \right), \quad |x| > 1.$$

### A.9 Complex transformation formulæ

$$\begin{aligned} \cos(ix) &= \cosh(x), & \cosh(ix) &= \cos(x), \\ \sin(ix) &= i \sinh(x), & \sinh(ix) &= i \sin x. \end{aligned}$$

### A.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with  $n$  factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. = 1 for the standard series) and *exponential like* (the radius of convergency is infinite).

**Power like:**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^n \binom{n}{j} x^j, \quad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \alpha \in \mathbb{R} \setminus \mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x| < 1,$$

$$\operatorname{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

**Exponential like:**

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R}.$$

### A.11 Magnitudes of functions

We often have to compare functions for  $x \rightarrow 0+$ , or for  $x \rightarrow \infty$ . The simplest type of functions are therefore arranged in an hierarchy:

- 1) logarithms,
- 2) power functions,
- 3) exponential functions,
- 4) faculty functions.

When  $x \rightarrow \infty$ , a function from a higher class will always dominate a function from a lower class. More precisely:

A) A *power function* dominates a *logarithm* for  $x \rightarrow \infty$ :

$$\frac{(\ln x)^\beta}{x^\alpha} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, \beta > 0.$$

B) An *exponential* dominates a *power function* for  $x \rightarrow \infty$ :

$$\frac{x^\alpha}{a^x} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, a > 1.$$

C) The *faculty function* dominates an *exponential* for  $n \rightarrow \infty$ :

$$\frac{a^n}{n!} \rightarrow 0, \quad n \rightarrow \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When  $x \rightarrow 0+$  we also have that a *power function* dominates the *logarithm*:

$$x^\alpha \ln x \rightarrow 0-, \quad \text{for } x \rightarrow 0+, \quad \alpha > 0.$$